1 Lagrange's Polynomial

* linear interpolation uses a line segment that passes throw two points.

Exp Find Lagrange polynomial through the points (x_0, y_0) and (x_1, y_1).

$$P_1(x) \approx f(x) = y = y_0 + m(x - x_0) \quad \text{where } m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$P_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$= y_0 + \frac{y_1}{x_1 - x_0}(x - x_0) - \frac{y_0}{x_1 - x_0}(x - x_0)$$

$$= y_0\left[1 - \frac{x - x_0}{x_1 - x_0}\right] + y_1 \frac{x - x_0}{x_1 - x_0}$$

$$= y_0 \frac{x_1 - x}{x_1 - x_0} + y_1 \frac{x - x_0}{x_1 - x_0}$$

Hence, the Lagrange polynomial of order 1 is

$$P_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

$$= y_0 L_{1,0}(x) + y_1 L_{1,1}(x)$$

$$= \sum_{k=0}^{1} y_k L_{1,k}(x)$$

Lagrange coefficient polynomials

• Note that $P_1(x_0) = y_0$ and $P_1(x_1) = y_1$

Remark: In general, the Lagrange Polynomial of degree at most n passes through $n+1$ points: $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ has the form:

$$P_n(x) = \sum_{k=0}^{n} y_k L_{n,k}(x)$$

where the Lagrange coefficient polynomial based on these points:

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Exp • Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

• Lagrange polynomial of order of degree ≤ 2 is

$$P_2(x) = \sum_{k=0}^{2} y_k L_{2,k}(x)$$

$$= y_0 L_{2,0}(x) + y_1 L_{2,1}(x) + y_2 L_{2,2}(x)$$

$$= y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Remark Note that $L_{n,k}(x) = \dfrac{\prod_{j=0, j\neq k}^{n} (x-x_j)}{\prod_{j=0, j\neq k}^{n} (x_k-x_j)}$ ✓

Exp* Given $\overset{x_0\ y_0}{(-1,6)}$, $\overset{x_1\ y_1}{(2,9)}$, $\overset{x_2\ y_2}{(0,3)}$. $\boxed{n=2}$

Find Lagrange polynomial and estimate $f(1)$

- $$P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= (6)\frac{(x-2)(x-0)}{(-1-2)(-1-0)} + (9)\frac{(x--1)(x-0)}{(2--1)(2-0)} + (3)\frac{(x--1)(x-2)}{(0--1)(0-2)}$$

$$= 2x(x-2) + \frac{3}{2}x(x+1) - \frac{3}{2}(x+1)(x-2)$$

$$= 2x^2 - 4x + \cancel{\frac{3}{2}x^2} + \frac{3}{2}x - \cancel{\frac{3}{2}x^2} + \frac{3}{2}x + 3$$

$$\boxed{P_2(x) = 2x^2 - x + 3}$$

"quadratic interpolation polynomial"

- To estimate $f(1)$, we use Lagrange polynomial:

$$f(1) \approx P_2(1) = 2 - 1 + 3 = 4$$

Exp Given $(x_0,y_0),(x_1,y_1),(x_2,y_2),(x_3,y_3) \Rightarrow P_3(x)$ is the Lagrange polynomial of degree ≤ 3 given by

$$P_3(x) = y_0 L_{3,0}(x) + y_1 L_{3,1}(x) + y_2 L_{3,2}(x) + y_3 L_{3,3}(x)$$

$$= y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} +$$

$$y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

"cubic interpolation polynomial"

Def (Uniform Partition)

- The partition of $[a,b] = [x_0, x_n]$ is uniform if the nodes $x_0, x_1, x_2, \dots, x_n$ are equally spaced.
- That is, $x_k = x_0 + hk$ for $k = 0, 1, 2, \dots, n$

Exp Consider $y = f(x) = \cos x$ on $[0, 1.2]$

1. Find Lagrange Polynomial of order 2 using equally spaced nodes. (use 3 digits)

- Nodes: $x_0 = 0$, $x_1 = 0.6$, $x_2 = 1.2$ which is $n+1 = 2+1 = 3$
- Points: (x_0, y_0), (x_1, y_1), (x_2, y_2)
  $(0, 1)$, $(0.6, 0.825)$, $(1.2, 0.362)$
- $$P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= (1)\frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)} + (0.825)\frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)} + (0.362)\frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)}$$

$$= 1.39(x-0.6)(x-1.2) - 2.29x(x-1.2) + 0.503x(x-0.6)$$

$$= -0.397x^2 + 0.246x + 0.698$$

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2. Find Lagrange Polynomial of order 3 using uniform partition.

- Nodes: $x_0 = 0$, $x_1 = 0.4$, $x_2 = 0.8$, $x_3 = 1.2$ which is $n+1 = 3+1 = 4$
- Points: (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)
  $(0, 1)$, $(0.4, 0.921)$, $(0.8, 0.697)$, $(1.2, 0.362)$

$$P_3(x) = y_0 L_{3,0}(x) + y_1 L_{3,1}(x) + y_2 L_{3,2}(x) + y_3 L_{3,3}(x)$$

$$= (1)\frac{(x-0.4)(x-0.8)(x-1.2)}{(0-0.4)(0-0.8)(0-1.2)} + (0.921)\frac{(x-0)(x-0.8)(x-1.2)}{(0.4-0)(0.4-0.8)(0.4-1.2)} +$$

$$(0.697)\frac{(x-0)(x-0.4)(x-1.2)}{(0.8-0)(0.8-0.4)(0.8-1.2)} + (0.362)\frac{(x-0)(x-0.4)(x-0.8)}{(1.2-0)(1.2-0.4)(1.2-0.8)}$$

$$= -2.60(x-0.4)(x-0.8)(x-1.2) + 7.20x(x-0.8)(x-1.2)$$

$$-5.44x(x-0.4)(x-1.2) + 0.944x(x-0.4)(x-0.8)$$

$$= 0.100x^3 - 0.580x^2 + 0.0300x + 0.998$$

Exp Let $f(x) = e^x$ on $[1, 4]$ (use 3 digits)

H.W

1. Find Lagrange Polynomial of order 1 using equally space nodes.
2. Find Lagrange Polynomial of order 2 using uniform partition.
3. Find Lagrange Polynomial of order 3 using uniform partition.

Remark Now we study the second method " Newton Polynomial " then we study the error term since the error of these two interpolations is the same.