Newton's Polynomial

Th (Newton's Polynomial)

- Given $x_0, x_1, x_2, \dots, x_n$ $n+1$ distinct numbers in $[a,b]$.
- Then, $\exists$ a unique polynomial $P_n(x)$ " Called Newton's Polynomial " of degree at most n s.t $f(x_i) = P_n(x_i)$ for $i = 1, 2, \dots, n$.
- Furthermore, Newton's Polynomial is given by

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \dots$$
$$+ a_n(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})$$

where the coefficients of Newton's Polynomial are given by the divided differences: $a_k = f[x_0, x_1, \dots, x_k]$ for $k = 0, 1, \dots, n$.

- That is:

$a_0 = f[x_0] = f(x_0) = y_0$: zero divided differences

$$a_1 = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} \quad \text{First divided differences}$$

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \quad 2^{nd} \text{ D.D}$$

$$= \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

$\vdots$

Divided Difference Table for $y = f(x)$

x_k	$f[x_k] = y_k$	1^{st} D.D	2^{nd} D.D	3^{rd} D.D
x_0	$y_0 = a_0$			
x_1	y_1	$f[x_0, x_1] = a_1$		
x_2	y_2	$f[x_1, x_2]$	$f[x_0, x_1, x_2] = a_2$	
x_3	y_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3] = a_3$
x_4	y_4	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$

Exp* Given $(\overset{x_0}{-1}, \overset{y_0}{6}), (\overset{x_1}{2}, \overset{y_1}{9}), (\overset{x_2}{0}, \overset{y_2}{3})$. Find Newton Polynomial.

- Newton Polynomial is $P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$

$$= 6 + a_1(x+1) + a_2(x+1)(x-2)$$

- $a_1 = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{9-6}{2+1} = 1$

$$a_2 = \frac{\dfrac{y_2 - y_1}{x_2 - x_1} - \dfrac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0} = \frac{\dfrac{3-9}{0-2} - \dfrac{9-6}{2+1}}{0+1} = 3 - 1 = 2$$

- Hence, $P_2(x) = 6 + x + 1 + 2(x+1)(x-2)$

$$= 2x^2 - x + 3$$

Exp Given $\overset{x_0}{-2}, \overset{y_0}{-12}$, i.e. $(x_0, y_0) = (-2,-12)$, $(x_1, y_1) = (-1,-4)$, $(x_2, y_2) = (1,0)$, $(x_3, y_3) = (2,8)$

- Construct the Divided Difference table
- Find Newton's Polynomial
- Estimate $f(0)$.

x_K	y_K	1^{st} D.D	2^{nd} D.D	3^{rd} D.D
-2	$a_0 = -12$			
-1	-4	$a_1 = 8$		
1	0	$f[x_1,x_2] = 2$	$a_2 = -2$	
2	8	$f[x_2,x_3] = 8$	$f[x_1,x_2,x_3] = 2$	$a_3 = 1$

$$f[x_2,x_3] = \frac{y_3 - y_2}{x_3 - x_2} = \frac{8-0}{2-1} = 8$$

- $a_0 = f[x_0] = y_0 = -12$

- $a_1 = f[x_0,x_1] = f[-2,-1] = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{-4+12}{-1+2} = 8$

- $a_2 = f[x_0,x_1,x_2] = f[-2,-1,1] = \dfrac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0} = \dfrac{\frac{0+4}{1+1} - 8}{1+2} = -2$

- $a_3 = f[x_0,x_1,x_2,x_3]$

$$= \frac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3 - x_0} = \frac{\frac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1} + 2}{2+2}$$

$$= \frac{\frac{8-2}{2+1} + 2}{4} = \frac{2+2}{4} = 1$$

- Newton's Polynomial is

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= -12 + 8(x+2) - 2(x+2)(x+1) + (x+2)(x+1)(x-1)$$

$$= x^3 + x - 2$$

- $f(0) \approx P_3(0) = 0 + 0 - 2 = -2$