Error Terms and Error Bounds

- **Error Term $E_n(x)$:**
  - Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
  - Given $P_n(x)$ interpolating Polynomial.

    → $P_n(x)$ can be Lagrange or Newton Polynomial that approximates $f(x)$

  - The Error term $E_n(x)$ is the same for Lagrange and Newton approximation $P_n(x)$
  - And $E_n(x)$ is similar to the error term for Taylor polynomial except the factor $(x - x_0)^{n+1}$ is replaced by the product $(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)$
  - This is because the interpolation $P_n(x)$ is exact " $P_n(x) = f(x)$ " at each $n+1$ nodes x_k ⇒

$$E_n(x_k) = f(x_k) - P_n(x_k) = y_k - y_k = 0$$

Th (Error Term)

- Assume $f \in C^{n+1}[a,b]$ and $x_0, x_1, \dots, x_n \in [a,b]$ are $n+1$ nodes.
- Then $f(x) = P_n(x) + E_n(x)$ where $E_n(x)$ is the error term given by $E_n(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)\, f^{(n+1)}(c)}{(n+1)!}$ for some $c = c(x)$ lies in $[a,b]$.

$P_n(x) = y_0 L_{n,0}(x) + y_1 L_{n,1}(x) + \cdots + y_n L_{n,n}(x)$ Lagrange Polynomial <u>or</u>

$P_n(x) = a_0 + a_1(x - x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$ Newton Polynomial

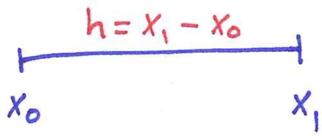
How to find an upper bound for the Error Term $E_n(x)$?

- That is, we need to find some constant s.t $|E_n(x)| \le$ constant.
- Finding the upper bound depends on whether the nodes are equally spaced (Uniform Partition) or not (not uniform partition).

Th (Upper Bound of the Error Term for Interpolation – Uniform Partition)

1. Given $(x_0, y_0), (x_1, y_1)$ "$n=1$"

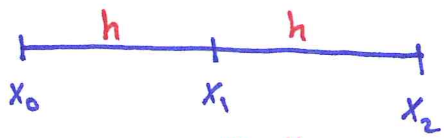
   with $E_1(x) = \dfrac{(x-x_0)(x-x_1)\, f''(c)}{2!}$

   Then $|E_1(x)| \le \dfrac{h^2 M_2}{8}$ where $M_2 = \displaystyle\max_{x_0 \le x \le x_1} |f''(x)|$

2. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ "$n=2$"

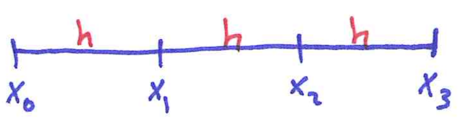
   with $E_2(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)\, f'''(c)}{3!}$

   Then $|E_2(x)| \le \dfrac{h^3 M_3}{9\sqrt{3}}$ where $M_3 = \displaystyle\max_{x_0 \le x \le x_2} |f'''(x)|$ $\quad h = \dfrac{x_2 - x_0}{2}$

3. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$ "$n=3$"

   with $E_3(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\, f^{(4)}(c)}{4!}$

   Then $|E_3(x)| \le \dfrac{h^4 M_4}{24}$ where $M_4 = \displaystyle\max_{x_0 \le x \le x_3} |f^{(4)}(x)|$ $\quad h = \dfrac{x_3 - x_0}{3}$

Exp Given $f(x) = \ln(x+2)$ on $[1, 1.6]$

Find an upper bound for E_1, E_2, E_3 using uniform partition.

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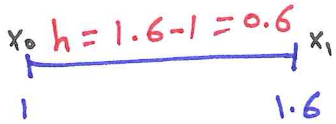
1. • $f'(x) = \dfrac{1}{x+2} \Rightarrow f''(x) = \dfrac{-1}{(x+2)^2} \Rightarrow |f''(x)| = \dfrac{1}{(x+2)^2}$

• The upper bound of E_1 is



$$|E_1(x)| \le \frac{h^2 M_2}{8} \quad \text{where} \quad M_2 = \underset{1 \le x \le 1.6}{\text{Max}} |f''(x)|$$

• $|f''|$ is decreasing $\Rightarrow |f''(x)| \le \dfrac{1}{(1+2)^2} = \dfrac{1}{9} = M_2$

• Hence, $|E_1(x)| \le \dfrac{(0.6)^2\left(\frac{1}{9}\right)}{8} = 0.005$

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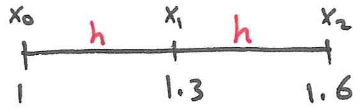
2. • The upper bound of E_2 is $|E_2(x)| \le \dfrac{h^3 M_3}{9\sqrt{3}}$

• $M_3 = \underset{1 \le x \le 1.6}{\text{Max}} |f'''(x)| \Rightarrow f'''(x) = \dfrac{2}{(x+2)^3}$



$h = \dfrac{1.6-1}{2} = 0.3$

• $|f'''(x)|$ is decreasing $\Rightarrow |f'''(x)| \le \left|\dfrac{2}{(1+2)^3}\right| = \dfrac{2}{27} = M_3$

• Hence, $|E_2(x)| \le \dfrac{(0.3)^3\left(\frac{2}{27}\right)}{9\sqrt{3}} = \dfrac{0.002}{9\sqrt{3}} = 0.000128$

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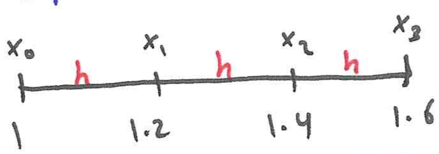
3. • The upper bound of E_3 is $|E_3(x)| \le \dfrac{h^4 M_4}{24}$

• $M_4 = \underset{1 \le x \le 1.6}{\text{Max}} |f^{(4)}(x)| \Rightarrow f^{(4)}(x) = \dfrac{-6}{(x+2)^4}$



$h = \dfrac{1.6-1}{3} = 0.2$

• $|f^{(4)}(x)| = \dfrac{6}{(x+2)^4} \le \dfrac{6}{(1+2)^4} = \dfrac{6}{81} = \dfrac{2}{27} = M_4$

• Hence, $|E_3(x)| \le \dfrac{(0.2)^4\left(\frac{2}{27}\right)}{24} = 0.00000494$

Upper Bound of the Error Term For interpolation - Non Uniform Partition

- Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ "$n=2$"

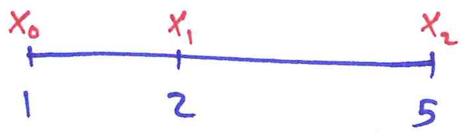
  with $E_2(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)\,f'''(c)}{3!}$

- Then $|E_2(x)| \le \dfrac{|\phi_2(x)|\;|f'''(c)|}{6}$ where

  $|\phi_2(x)|$ is an upper bound of $\phi_2(x) = (x-x_0)(x-x_1)(x-x_2)$ and
  $|f'''(c)|$ is an upper bound of $f'''(x)$.

Exp Given $f(x) = x^2 - \dfrac{2}{x}$ and $(1, f(1)), (2, f(2)), (5, f(5))$.
Find an upper bound for the error term of interpolation.

- $n=2$



Not uniform

- The upper bound of error is

$$|E_2(x)| \le \frac{|\phi_2(x)|\;|f'''(c)|}{6}$$

- $f'(x) = 2x + \dfrac{2}{x^2} \Rightarrow f''(x) = 2 - \dfrac{4}{x^3} \Rightarrow$

  $f'''(x) = \dfrac{12}{x^4} \Rightarrow |f'''(x)| = \dfrac{12}{x^4} \le 12 = M_3$

- $\phi_2(x) = (x-1)(x-2)(x-5) = (x^2-3x+2)(x-5)$

  $\phi_2'(x) = (x^2-3x+2)(1) + (x-5)(2x-3) = 3x^2 - 16x + 17 = 0$

  $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{16 \pm \sqrt{(16)^2 - 4(3)(7)}}{6} \Rightarrow x_1 = 3.8685$ or $x_2 = 1.4682$

  $|\phi(x_1)| = \boxed{6.06}$ Max and $|\phi(x_2)| = 3.783$, $|\phi(1)| = |\phi(5)| = 0$ end points

- Hence, $|E_2(x)| \le \dfrac{(6.06)(12)}{6} = 12.12$

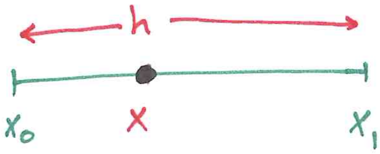
Proof of Th$^{\$}$

$\boxed{\text{I}}$ $|E_1(x)| \leq \dfrac{h^2 M_2}{8}$ where $M_2 = \underset{x_0 \leq x \leq x_1}{\text{Max}} |f''(x)|$

- The Error Term is $E_1(x) = \dfrac{(x-x_0)(x-x_1) f''(c)}{2} \Rightarrow$

$$|E_1(x)| = \frac{|\phi_1(x)|\,|f''(c)|}{2} \quad \text{where } |f''(c)| \leq M_2$$

- Now we need to find an upper bound for $|\phi_1(x)|$.
  - $\phi_1(x) = (x-x_0)(x-x_1)$ using change of variables



  - Let $\boxed{x - x_0 = t}$
  - We have

$$x_1 = x_0 + h \qquad\qquad x_0 \leq x \leq x_1$$
$$-x_1 = -x_0 - h$$
$$x - x_1 = x - x_0 - h \qquad\qquad 0 \leq x - x_0 \leq x_1 - x_0$$
$$\boxed{x - x_1 = t - h} \qquad\qquad \boxed{0 \leq t \leq h}$$

- $\phi_1(x) = \phi_1(x_0 + t) = t(t-h) = t^2 - ht = \phi_1(t)$

$$\phi_1' = 2t - h = 0 \iff t = \frac{h}{2} \text{ critical point}$$

- $\left|\phi\left(\frac{h}{2}\right)\right| = \left|\frac{h^2}{4} - \frac{h^2}{2}\right| = \boxed{\dfrac{h^2}{4}}$ Max since $|\phi(0)| = |\phi(h)| = 0$ end points

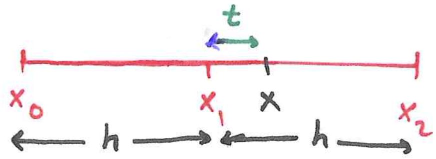
- Hence, $|E_1(x)| = \dfrac{|\phi_1(x)|\,|f''(c)|}{2} \leq \dfrac{\frac{h^2}{4} M_2}{2} = \dfrac{h^2 M_2}{8}$

$\boxed{2}$ $\quad |E_2(x)| \le \dfrac{h^3 M_3}{9\sqrt{3}}$ where $M_3 = \displaystyle\max_{x_0 \le x \le x_2} |f'''(x)|$

- The Error Term is $E_2(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)\, f'''(c)}{3!} \Rightarrow$

$$|E_2(x)| = \frac{|\phi_2(x)|\,|f'''(c)|}{6} \quad \text{where } |f'''(c)| \le M_3$$

- Now we need to find an upper bound for $|\phi_2(x)|$.
  - $\phi_2(x) = (x-x_0)(x-x_1)(x-x_2)$



  - Using the change of variable:

$$x - x_1 = t$$
$$x - x_0 = t + h$$
$$x - x_2 = t - h$$

$$x_0 \le x \le x_2$$
$$x_0 - x_1 \le x - x_1 \le x_2 - x_1$$
$$-h \le t \le h$$

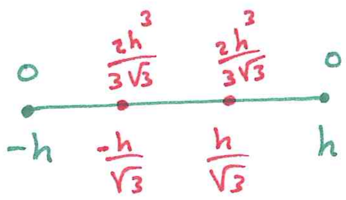
- $\phi_2(x) = \phi_2(x_1 + t) = (t+h)(t)(t-h)$

$$= t(t^2 - h^2)$$
$$= t^3 - t h^2$$
$$= \phi(t)$$

$$\phi_2' = 3t^2 - h^2 = 0 \iff t = \pm \frac{h}{\sqrt{3}} \quad \text{critical points}$$

- $\left|\phi_2\left(\frac{h}{\sqrt{3}}\right)\right| = \left|\dfrac{h^3}{3\sqrt{3}} - \dfrac{h^3}{\sqrt{3}}\right| = \dfrac{2h^3}{3\sqrt{3}}$

$$\left|\phi_2\left(\frac{-h}{\sqrt{3}}\right)\right| = \left|\frac{-h^3}{3\sqrt{3}} + \frac{h^3}{\sqrt{3}}\right| = \frac{2h^3}{3\sqrt{3}}$$



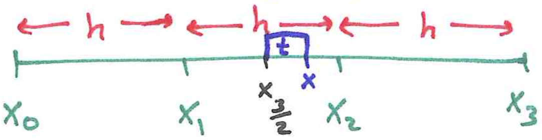
- Hence, $|E_2(x)| = \dfrac{|\phi_2(x)|\,|f'''(c)|}{6} \le \dfrac{\frac{2h^3}{3\sqrt{3}} M_3}{6} = \dfrac{h^3 M_3}{9\sqrt{3}}$

$\boxed{3}$ $|E_3(x)| \leq \dfrac{h^4 M_4}{24}$ where $M_4 = \underset{x_0 \leq x \leq x_3}{\text{Max}} |f^{(4)}(x)|$

- The Error Term is $E_3(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\, f^{(4)}(c)}{4!} \Rightarrow$

$$|E_3(x)| = \frac{|\phi_3(x)|\,|f^{(4)}(c)|}{24} \quad \text{where} \quad |f^{(4)}(c)| \leq M_4$$

- Now we need to find an upper bound for $|\phi_3(x)|$.
  - $\phi_3(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)$



  - Using the change of variable:

$$x - x_0 = t + \tfrac{3}{2}h$$
$$x - x_1 = t + \tfrac{h}{2}$$
$$x - x_2 = t - \tfrac{h}{2}$$
$$x - x_3 = t - \tfrac{3}{2}h$$

$$x_{\frac{3}{2}} = x_0 + \tfrac{3}{2}h$$

$$x_0 \leq x \leq x_3$$

$$x_0 - x_{\frac{3}{2}} \leq x - x_{\frac{3}{2}} \leq x_3 - x_{\frac{3}{2}}$$

$$-\tfrac{3}{2}h \leq t \leq \tfrac{3}{2}h$$

- $\phi_3(x) = (t + \tfrac{3}{2}h)(t + \tfrac{h}{2})(t - \tfrac{h}{2})(t - \tfrac{3}{2}h)$
$$= (t^2 - \tfrac{9}{4}h)(t^2 - \tfrac{h^2}{4}) = t^4 - \tfrac{5}{2}h^2t^2 + \frac{9h^4}{16} = \phi_3(t)$$

- $\phi_3' = 4t^3 - 5h^2 t = 0 \Leftrightarrow t(4t^2 - 5h^2) = 0 \Leftrightarrow t = 0$ or $t = \pm \frac{\sqrt{5}}{2}h$

- $|\phi_3(0)| = \dfrac{9h^4}{16}$, $|\phi_3(\pm \frac{\sqrt{5}}{2}h)| = h^4$ Max

- Hence, $|E_3(x)| = \dfrac{|\phi_3(x)|\,|f^{(4)}(c)|}{24}$

$$\leq \frac{h^4 M_4}{24}$$