Linearization

Exp • Given the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

• Find the least-squar exponential curve of the form

$$y = C e^{Dx}$$

• $E(C, D) = \sum_{i=1}^{n} \left(C e^{Dx_i} - y_i \right)^2$

• $\frac{\partial E}{\partial C} = 2 \sum_{i=1}^{n} \left(\left(C e^{Dx_i} - y_i \right) e^{Dx_i} \right) = 0 \iff$

$$\boxed{C \sum_{i=1}^{n} e^{2Dx_i} - \sum_{i=1}^{n} y_i e^{Dx_i} = 0} \quad ①$$

• $\frac{\partial E}{\partial D} = 2 \sum_{i=1}^{n} \left(C e^{Dx_i} - y_i \right) \cancel{C} x_i e^{Dx_i} = 0 \iff$

$$\boxed{C \sum_{i=1}^{n} x_i e^{2Dx_i} - \sum_{i=1}^{n} y_i x_i e^{Dx_i} = 0} \quad ②$$

• The normal equations ① and ② are hard to solve and find C and D.

• So we use a technique called linearization.

- linearization for $y = c\,e^{Dx}$ works like this:
  - Take logarithm of both sides:

$$\ln y = Dx + \ln c$$

  - Then introduce the change of variables:

$$Y = Dx + E \quad \text{where } Y = \ln y, \quad E = \ln c$$

  - Now use the linear normal equations page 109

$$D \sum_{i=1}^{n} x_i^2 + E \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i Y_i$$

$$D \sum_{i=1}^{n} x_i + nE = \sum_{i=1}^{n} Y_i \qquad *$$

- Solve these equations for D and E $\Rightarrow$

Then $c = e^{E}$ and so $y = f(x) = c\,e^{Dx}$

Exp Find the exponential fit $y = c\,e^{Dx}$ using linearization for the following five data points:

$(0, 1.5), (1, 2.5), (2, 3.5), (3, 5), (4, 7.5)$

- First we solve the linear normal equations $*$ and find the constants D and E
- Then we find $c = e^{E}$
- Hence, $y = f(x) = c\,e^{Dx}$

x_i	y_i	x_i^2	$Y_i = \ln y_i$	$x_i Y_i$
0	1.5	0	0.405465	0
1	2.5	1	0.916291	0.916291
2	3.5	4	1.252763	2.505526
3	5	9	1.609438	4.828314
4	7.5	16	2.014903	8.059612
Total		30	6.198860	16.309743

- The linear normal equations become:

$$30\,D + 10\,E = 16.309743$$
$$10\,D + 5\,E = 6.198860$$

- The solution is $D = 0.3912023$ and $E = 0.457367$
- Now we find $C = e^{E} = e^{0.457367} = 1.579910$
- Hence, $y = f(x) = C\,e^{Dx}$

$$= 1.579910\, e^{0.3912023\,x}$$

The exponential fit $y = 1.579910\,e^{0.3912023x}$ obtained by using the linearization method.