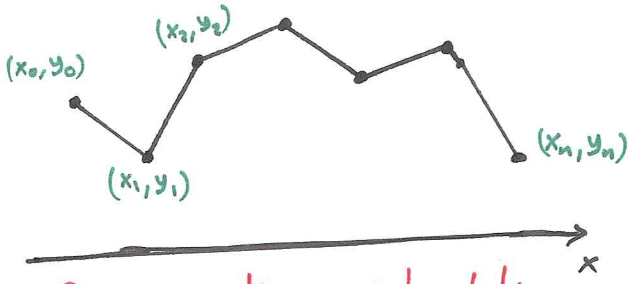
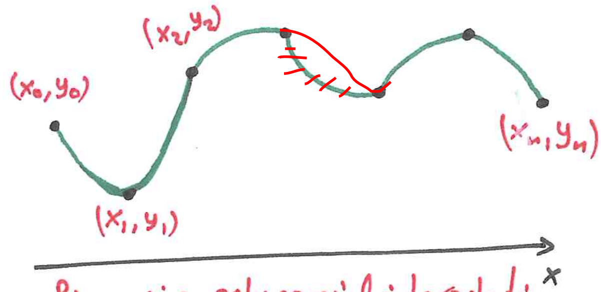
5.3 Interpolation by Spline Functions

- In this section we study a piecewise interpolation.
- Piecewise interpolation can be linear or nonlinear "polynomial" interpolation.



Piecewise linear interpolation
" linear spline "



Piecewise polynomial interpolation
" Cubic Spline "

Def (Piecewise linear spline)

- The piecewise linear curve defined on $[x_k, x_{k+1}]$ is

$$S_k(x) = y_k + d_k(x - x_k)$$

where $k = 0, 1, 2, \dots, n-1$ and $d_k = \dfrac{y_{k+1} - y_k}{x_{k+1} - x_k}$.

- That is :

$$S(x) = \begin{cases} S_0(x) = y_0 + d_0(x - x_0), & x_0 \le x \le x_1 \\ S_1(x) = y_1 + d_1(x - x_1), & x_1 \le x \le x_2 \\ \vdots & \vdots \\ S_k(x) = y_k + d_k(x - x_k), & x_k \le x \le x_{k+1} \\ \vdots & \vdots \\ S_{n-1}(x) = y_{n-1} + d_{n-1}(x - x_{n-1}), & x_{n-1} \le x \le x_n \end{cases}$$

Remark : The Lagrange polynomial is used to represent this piecewise linear spline :

$$S_k(x) = y_k \frac{x - x_{k+1}}{x_k - x_{k+1}} + y_{k+1} \frac{x - x_k}{x_{k+1} - x_k} \quad \text{for } x_k \le x \le x_{k+1}$$

where $k = 0, 1, 2, \dots, n-1$

Def (Piecewise Cubic Splines)

- Given $n+1$ points: $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
- The function $S(x)$ defined by n formula on $[a,b]=[x_0, x_n]$:

$$S(x) = \begin{cases} S_0(x) = A_0(x-x_0)^3 + B_0(x-x_0)^2 + C_0(x-x_0) + D_0, & x_0 \le x \le x_1 \\ S_1(x) = A_1(x-x_1)^3 + B_1(x-x_1)^2 + C_1(x-x_1) + D_1, & x_1 \le x \le x_2 \\ \vdots & \vdots \\ S_{n-1}(x) = A_{n-1}(x-x_{n-1})^3 + B_{n-1}(x-x_{n-1})^2 + C_{n-1}(x-x_{n-1}) + D_{n-1}, & x_{n-1} \le x \le x_n \end{cases}$$

is called cubic spline iff the following conditions hold:

$\boxed{1}$
$$\begin{aligned} S_0(x_0) &= y_0 \\ S_1(x_1) &= y_1 \\ S_2(x_2) &= y_2 \\ &\vdots \\ S_{n-1}(x_{n-1}) &= y_{n-1} \\ S_{n-1}(x_n) &= y_n \end{aligned}$$

$n+1$ conditions (equations)

$\boxed{2}$
$$\begin{aligned} S_0(x_1) &= S_1(x_1) \\ S_1(x_2) &= S_2(x_2) \\ &\vdots \\ S_{n-2}(x_{n-1}) &= S_{n-1}(x_{n-1}) \end{aligned}$$

$n-1$ conditions

$\boxed{3}$
$$\begin{aligned} S_0'(x_1) &= S_1'(x_1) \\ S_1'(x_2) &= S_2'(x_2) \\ &\vdots \\ S_{n-2}'(x_{n-1}) &= S_{n-1}'(x_{n-1}) \end{aligned}$$

$n-1$ conditions

$\boxed{4}$
$$\begin{aligned} S_0''(x_1) &= S_1''(x_1) \\ S_1''(x_2) &= S_2''(x_2) \\ &\vdots \\ S_{n-2}''(x_{n-1}) &= S_{n-1}''(x_{n-1}) \end{aligned}$$

$n-1$ conditions

Remark: we use cubic splines to estimate $f(x)$ on $[a,b]=[x_0,x_n]$

③ Cubic splines produces $4n-2$ equations but we have $4n$ unknows so there is two degree of freedom (2 missing conditions).

② We use cubic splines to estimate $f(x)$ because we can make its first and second derivatives all continuous on the large interval $[x_0, x_n]$ so that $S(x)=y$ has no sharp corners.

- Depending on the remaining two conditions, there are two types of cubic spline:

1. Clamped Cubic Spline: $S'(a) = S_0'(x_0) = f'(x_0)$

   $S'(b) = S_{n-1}'(x_n) = f'(x_n)$

2. Natural Cubic Spline: $S''(a) = S_0''(x_0) = 0$

   $S''(b) = S_{n-1}''(x_n) = 0$

Exp Given (x_0, y_0) and (x_1, y_1). Write the form of the cubic spline $S(x)$ that estimat $y = f(x)$. $n=1$

$$S(x) = S_0(x) = A_0(x-x_0)^3 + B_0(x-x_0)^2 + C_0(x-x_0) + D_0, \quad x_0 \le x \le x_1$$

since $n = 1$

Exp Given the following data points: $(1,2), (2,3), (3,5)$. $n=2$ (x_0, x_1, x_2)

i. Find the natural cubic spline through these data.

ii. Find the clamped cubic spline through these data given that $f'(1) = 2$ and $f'(3) = 1$

- $n = 2 \Rightarrow$

$$S(x) = \begin{cases} S_0(x) = A_0(x-1)^3 + B_0(x-1)^2 + C_0(x-1) + D_0, & 1 \le x \le 2 \\ S_1(x) = A_1(x-2)^3 + B_1(x-2)^2 + C_1(x-2) + D_1, & 2 \le x \le 3 \end{cases}$$

$$S'(x) = \begin{cases} S_0'(x) = 3A_0(x-1)^2 + 2B_0(x-1) + C_0 \quad , \quad 1 \le x \le 2 \\ S_1'(x) = 3A_1(x-2)^2 + 2B_1(x-2) + C_1 \quad , \quad 2 \le x \le 3 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 6A_0(x-1) + 2B_0 \quad , \quad 1 \le x \le 2 \\ S_1''(x) = 6A_1(x-2) + 2B_1 \quad , \quad 2 \le x \le 3 \end{cases}$$

$\boxed{1} \Rightarrow \quad S_0(x_0) = y_0 \Leftrightarrow S_0(1) = 2 \Leftrightarrow \boxed{D_0 = 2}$

$S_1(x_1) = y_1 \Leftrightarrow S_1(2) = 3 \Leftrightarrow \boxed{D_1 = 3}$

$S_1(x_2) = y_2 \Leftrightarrow S_1(3) = 5 \Leftrightarrow A_1 + B_1 + C_1 + D_1 = 5$

$\Leftrightarrow \boxed{A_1 + B_1 + C_1 = 2} - *^1$

$\boxed{2} \Rightarrow \quad S_0(x_1) = S_1(x_1) \Leftrightarrow S_0(2) = S_1(2) \Leftrightarrow A_0 + B_0 + C_0 + D_0 = D_1$

$\Leftrightarrow \boxed{A_0 + B_0 + C_0 = 1} - *^2$

$\boxed{3} \Rightarrow \quad S_0'(x_1) = S_1'(x_1) \Leftrightarrow S_0'(2) = S_1'(2) \Leftrightarrow \boxed{3A_0 + 2B_0 + C_0 = C_1} - *^3$

$\boxed{4} \Rightarrow \quad S_0''(x_1) = S_1''(x_1) \Leftrightarrow S_0''(2) = S_1''(2) \Leftrightarrow \boxed{6A_0 + 2B_0 = 2B_1} - *^4$

$\boxed{i}$ For Natural Cubic spline $\Rightarrow$

$S''(a) = S_0''(x_0) = S_0''(1) = 0 \Leftrightarrow 2B_0 = 0 \Leftrightarrow \boxed{B_0 = 0}$

$S''(b) = S_1''(x_2) = S_1''(3) = 0 \Leftrightarrow 6A_1 + 2B_1 = 0 \Leftrightarrow \boxed{3A_1 + B_1 = 0} - *^5$

$*^2 \Rightarrow C_0 = 1 - A_0$ so $*^3$ becomes $3A_0 + 1 - A_0 = C_1$

$$\Downarrow$$

$$2A_0 + 1 = C_1$$

$*^4 \Rightarrow B_1 = 3A_0$ so $*^1$ becomes $A_1 + 3A_0 + 2A_0 + 1 = 2$

$$\Downarrow$$

$$A_1 + 5A_0 = 1$$

$*^5$ becomes $A_1 + A_0 = 0$

$$\boxed{A_0 = \tfrac{1}{4}} \quad \boxed{A_1 = \tfrac{-1}{4}} \quad \boxed{C_0 = \tfrac{3}{4}} \quad \boxed{B_1 = \tfrac{3}{4}} \quad \boxed{C_1 = \tfrac{3}{2}}$$

Hence, the natural cubic spline is

$$S(x) = \begin{cases} S_0(x) = \frac{1}{4}(x-1)^3 + \frac{3}{4}(x-1) + 2, & 1 \le x \le 2 \\ S_1(x) = \frac{-1}{4}(x-2)^3 + \frac{3}{4}(x-2)^2 + \frac{3}{2}(x-2) + 3, & 2 \le x \le 3 \end{cases}$$

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ii For clamped cubic spline $\Rightarrow$

- $S'(a) = S'_0(x_0) = S'_0(1) = f'(1) = 2 \Leftrightarrow S'_0(1) = 2 \Leftrightarrow \boxed{C_0 = 2}$

  $S'(b) = S'_1(x_2) = S'_1(3) = f'(3) = 1 \Leftrightarrow S'_1(3) = 1 \Leftrightarrow \boxed{3A_1 + 2B_1 + C_1 = 1}$ $*^6$
- solving $*^1, *^2, *^3, *^4, *^6$ gives $A_0 = \frac{5}{2}$, $A_1 = \frac{-5}{2}$, $B_0 = \frac{-7}{2}$, $B_1 = 4$, $C_1 = \frac{1}{2}$
- Hence, the clamped cubic spline is

$$S(x) = \begin{cases} S_0(x) = \frac{5}{2}(x-1)^3 - \frac{7}{2}(x-1)^2 + 2(x-1) + 2, & 1 \le x \le 2 \\ S_1(x) = -\frac{5}{2}(x-2)^3 + 4(x-2)^2 + \frac{1}{2}(x-2) + 3, & 2 \le x \le 3 \end{cases}$$

out

Exp • Given $\overset{x_0\,y_0}{(0,1)}, \overset{x_1\,y_1}{(1,2)}, \overset{x_2\,y_2}{(3,4)}$

• Find the Natural Cubic Spline

• $n=2$ $\Rightarrow$ The cubic Spline is

$$S(x) = \begin{cases} S_0(x) = A_0 x^3 + B_0 x^2 + C_0 x + D_0 , & 0 \le x \le 1 \\ S_1(x) = A_1 (x-1)^3 + B_1(x-1)^2 + C_1(x-1) + D_1 , & 1 \le x \le 3 \end{cases}$$

$$S'(x) = \begin{cases} S_0'(x) = 3A_0 x^2 + 2B_0 x + C_0 , & 0 \le x \le 1 \\ S_1'(x) = 3A_1 (x-1)^2 + 2B_1(x-1) + C_1 , & 1 \le x \le 3 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 6A_0 x + 2B_0 , & 0 \le x \le 1 \\ S_1''(x) = 6A_1 (x-1) + 2B_1 , & 1 \le x \le 3 \end{cases}$$

1. $S_0(x_0) = y_0 \Leftrightarrow S_0(0) = 1 \Leftrightarrow \boxed{D_0 = 1}$

   $S_1(x_1) = y_1 \Leftrightarrow S_1(1) = 2 \Leftrightarrow \boxed{D_1 = 2}$

   $S_1(x_2) = y_2 \Leftrightarrow S_1(3) = 4 \Leftrightarrow 8A_1 + 4B_1 + 2C_1 + 2 = 4$

   $\Leftrightarrow \boxed{4A_1 + 2B_1 + C_1 = 1} - *^1$

2. $S_0(x_1) = S_1(x_1) \Leftrightarrow S_0(1) = S_1(1)$

   $\Leftrightarrow A_0 + B_0 + C_0 + 1 = 2$

   $\Leftrightarrow \boxed{A_0 + B_0 + C_0 = 1} - *^2$

3. $S_0'(x_1) = S_1'(x_0) \Leftrightarrow S_0'(1) = S_1'(1) \Leftrightarrow \boxed{3A_0 + 2B_0 + C_0 = C_1} - *^3$

4. $S_0''(x_1) = S_1''(x_0) \Leftrightarrow S_0''(1) = S_1''(1) \Leftrightarrow 6A_0 + 2B_0 = 2B_1$

   $\Leftrightarrow \boxed{3A_0 + B_0 = B_1} - *^4$

For natural cubic spline $\Rightarrow$

$$S''(a) = S_0''(x_0) = S_0''(0) = 0 \iff \boxed{B_0 = 0}$$

$$S''(b) = S_1''(x_2) = S_1''(3) = 0 \iff 12A_1 + 2B_1 = 0$$

$$\iff \boxed{B_1 = -6A_1} - *^5$$

- Solving $*^1, *^2, *^3, *^4, *^5$ gives $\boxed{A_0 = A_1 = B_1 = 0}$ and $\boxed{C_0 = C_1 = 1}$
- Hence, the natural cubic spline becomes linear:

$$S(x) = \begin{cases} S_0(x) = x + 1, & 0 \le x \le 1 \\ S_1(x) = x+1, & 1 \le x \le 3 \end{cases}$$

$$= x + 1 \quad \text{on} \quad 0 \le x \le 3$$

Exp Consider the following function:

$$S(x) = \begin{cases} S_0(x) = x^3 + x - 1 & , \ 0 \leq x \leq 1 \\ S_1(x) = 1 + C(x-1) + D(x-1)^2 - (x-1)^3 & , \ 1 \leq x \leq 2 \end{cases}$$

[a] Find the constats C and D that makes $S(x)$ cubic spline.

[b] Is $S(x)$ natural cubic spline?

$$S'(x) = \begin{cases} S_0'(x) = 3x^2 + 1 & , \ 0 \leq x \leq 1 \\ S_1'(x) = C + 2D(x-1) - 3(x-1)^2 & , \ 1 \leq x \leq 2 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 6x & , \ 0 \leq x \leq 1 \\ S_1''(x) = 2D - 6(x-1) & , \ 1 \leq x \leq 2 \end{cases}$$

[a] • $S(x)$ is continuous at $x_1 = 1$ $\Leftrightarrow S_0(1) = S_1(1)$

$\Leftrightarrow 1 = 1$ does not helpe

• $S(x)$ is differentiable at $x_1 = 1$ $\Leftrightarrow S_0'(1) = S_1'(1)$

$\Leftrightarrow \boxed{4 = C}$

• $S(x)$ is twice diff. at $x_1 = 1$ $\Leftrightarrow S_0''(1) = S_1''(1)$

$\Leftrightarrow 6 = 2D \Leftrightarrow \boxed{D = 3}$

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[b] We check if $S_0''(x_0) = S_0''(0) \overset{?}{=} 0 \Rightarrow S_0''(0) = (6)(0) = 0$

and $S_1''(x_2) = S_1''(2) \overset{?}{=} 0 \Rightarrow S_1''(2) = 2(3) - 6(2-1) = 0$

so the cubic spline $S(x)$ is natural.