

D.F for $f'(x_0)$

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[1] C.D.F of order $o(h^2)$:

$$\begin{aligned} f'(x_0) &\approx \frac{f(x_0+h) - f(x_0-h)}{2h} + E_{\text{trun}}(f, h) \\ &= \frac{f_1 - f_{-1}}{2h} + \frac{-h^2 f'''(c)}{6} \end{aligned}$$

[2] F.D.F of order $o(h^2)$:

$$f'(x_0) \approx \frac{-3f_0 + 4f_1 - f_2}{2h} + \frac{h^2 f'''(c)}{3}$$

[3] B.D.F of order $o(h^2)$

$$f'(x_0) \approx \frac{3f_0 - 4f_1 + f_{-2}}{2h} + \frac{h^2 f'''(c)}{3}$$

[4] C.D.F of order $o(h^4)$

$$f'(x_0) \approx \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

Remark • $E_{\text{trun}}(f, h)$ is called the truncation error.

- In case of [1], [2], [3] $\Rightarrow f$ is assumed to be $C^3[a, b]$
- In case of [4] $\Rightarrow f$ is assumed to be $C^5[a, b]$
- And so on ...

D.F for $f''(x_0)$

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① C.D.F of order $o(h^2)$:

$$f''(x_0) \approx \frac{f_1 - 2f_0 + f_{-1}}{h^2} + \frac{-h^2 f^{(4)}(c)}{12}$$

② F.D.F of order $o(h^2)$:

$$f''(x_0) \approx \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} + \frac{11h^2 f^{(4)}(c)}{12}$$

③ B.D.F of order $o(h^2)$

$$f''(x_0) \approx \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2} + \frac{11h^2 f^{(4)}(c)}{12}$$

④ C.D.F of order $o(h^4)$:

$$f''(x_0) = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} + \frac{h^4 f^{(6)}(c)}{90}$$

Remark

To derive any of these D.F's for $f'(x_0)$ and $f''(x_0)$, we use:

- ① Taylor's Expansion or
- ② Lagrange Interpolation or
- ③ Newton Interpolation.

Exp Consider the following points:

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$(2, -1), (2.1, 2), (2.2, -1.5), (2.3, 0), (2.4, 2)$

(1) Estimate $f'(2.1)$ using C.D.F of order $O(h^2)$

(2) Estimate $f'(2.2)$ using F.D.F of order $O(h^2)$

clearly $h = 0.1$

$$\begin{aligned} \text{(1)} \quad f'(2.1) &\approx \frac{f_1 - f_{-1}}{2h} = \frac{f(2.1+0.1) - f(2.1-0.1)}{2(0.1)} \\ &= \frac{f(2.2) - f(2)}{0.2} = \frac{-1.5 - (-1)}{0.2} = \frac{-0.5}{0.2} = -2.5 \end{aligned}$$

$$\begin{aligned} \text{(2)} \quad f'(2.2) &\approx \frac{-3f_0 + 4f_1 - f_2}{2h} = \frac{-3f(2.2) + 4f(2.3) - f(2.4)}{2(0.1)} \\ &= \frac{-3(-1.5) + 4(0) - 2}{0.2} = \frac{2.5}{0.2} = 12.5 \end{aligned}$$

Exp
out Let $f(x) = e^x$. Estimate $f'(1)$ using B.D.F of order $O(h^2)$ with (i) $h = 0.01$ (ii) $h = 0.001$

$$f'(1) \approx \frac{3f_0 - 4f_1 + f_2}{2h} = \frac{3f(1) - 4f(1-h) + f(1-2h)}{2h}$$

$$\text{(i)} \quad f'(1) = \frac{3f(1) - 4f(0.99) + f(0.98)}{0.02} = \frac{3e - 4e^{0.99} + e^{0.98}}{0.02}$$

$$= 2.7181918955$$

$$\text{(ii)} \quad f'(1) = \frac{3f(1) - 4f(0.999) + f(0.998)}{0.002} = \frac{3e - 4e^{0.999} + e^{0.998}}{0.002}$$

$$= 2.718280923$$

Exp Let $f(x) = \sin x$

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① Estimate $f'(0)$ using C.D.F of order $O(h^4)$ with step size $h = 0.01$

② Estimate $f'(3)$ using B.D.F of order $O((0.1)^2)$.

$$\begin{aligned} \text{① } f'(0) &\approx \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} \\ &= \frac{-f(0.02) + 8f(0.01) - 8f(-0.01) + f(-0.02)}{12(0.01)} \\ &= \frac{-2\sin(0.02) + 16\sin 0.01}{0.12} \approx \frac{0.120061199}{0.12} = 1.00051 \end{aligned}$$

$$\begin{aligned} \text{② } h = 0.1 \Rightarrow f'(3) &\approx \frac{3f_0 - 4f_{-1} + f_{-2}}{2h} \\ x_0 = 3 & \\ &= \frac{3f(3) - 4f(2.9) + f(2.8)}{2(0.1)} \\ &= \frac{3\sin 3 - 4\sin(2.9) + \sin(2.8)}{0.2} \\ &= -0.9939715075 \end{aligned}$$

Note that the true values are:

$$f'(x) = \cos x$$

$$f'(0) = \cos 0 = 1$$

$$f'(3) = \cos 3 = -0.9902072488$$

Exp Consider the following table:

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t	1	1.3	1.6	1.9
D	10	30	60	100

where t: time

D: distance

- ① Estimate the velocity at $t=1.6$ using C.D.F of $O(h^2)$
- ② Estimate the acceleration at $t=1.3$ using C.D.F of $O(h^2)$
- ③ Estimate the velocity at $t=1.6$ using F.D.F of $O(h^2)$.

$$h = 0.3 \Rightarrow$$

$$\begin{aligned} \text{① } V(t_0) = \dot{D}(t_0) &\approx \frac{D_1 - D_{-1}}{2h} = \frac{D(t_0+h) - D(t_0-h)}{2h} \\ t_0 = 1.6 & \\ \dot{D}(1.6) &\approx \frac{D(1.6+0.3) - D(1.6-0.3)}{2(0.3)} = \frac{D(1.9) - D(1.3)}{0.6} \\ &= \frac{100 - 30}{0.6} = \frac{70}{0.6} = 116.67 \end{aligned}$$

$$\begin{aligned} \text{② } a(t_0) = \ddot{D}(t_0) &\approx \frac{D_1 - 2D_0 + D_{-1}}{h^2} = \frac{D(1.6) - 2D(1.3) + D(1)}{(0.3)^2} \\ t_0 = 1.3 & \\ &= \frac{60 - 2(30) + 10}{0.09} = 111.11 \end{aligned}$$

$$\begin{aligned} \text{③ } V(t_0) = \dot{D}(t_0) &\approx \frac{-3\overset{\checkmark}{D_0} + 4\overset{\checkmark}{D_1} - \overset{\times}{D_2}}{2h} \quad \text{not possible} \\ t_0 = 1.6 & \end{aligned}$$

D.F for $f'(x_0)$

① C.D.F of order $o(h^2)$: $f'(x_0) = \frac{f_1 - f_{-1}}{2h} - \frac{h^2 f'''(c)}{6}$

② F.D.F of order $o(h^2)$: $f'(x_0) = \frac{-3f_0 + 4f_1 - f_2}{2h} + \frac{h^2 f'''(c)}{3}$

③ B.D.F of order $o(h^2)$: $f'(x_0) = \frac{3f_0 - 4f_{-1} + f_{-2}}{2h} + \frac{h^2 f'''(c)}{3}$

④ C.D.F of order $o(h^4)$: $f'(x_0) = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$

D.F for $f''(x_0)$

① C.D.F of order $o(h^2)$: $f''(x_0) = \frac{f_1 - 2f_0 + f_{-1}}{h^2} - \frac{h^2 f^{(4)}(c)}{12}$

② F.D.F of order $o(h^2)$: $f''(x_0) = \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} + \frac{11h^2 f^{(4)}(c)}{12}$

③ B.D.F of order $o(h^2)$: $f''(x_0) = \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2} + \frac{11h^2 f^{(4)}(c)}{12}$

④ C.D.F of order $o(h^4)$: $f''(x_0) = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} + \frac{h^4 f^{(6)}(c)}{90}$