Derivation of D.F's

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- Recall Taylor's expansion of $f(x)$ about x_0:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

- Clearly $\Rightarrow$ when $x = x_0 + h \Rightarrow x - x_0 = h \Rightarrow$

$$f_1 = f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots$$

$$f_{-1} = f(x_0-h) = f(x_0) - h f'(x_0) + \frac{h^2}{2!} f''(x_0) - \frac{h^3}{3!} f'''(x_0) + \cdots$$

$$f_2 = f(x_0+2h) = f(x_0) + 2h f'(x_0) + \frac{(2h)^2}{2!} f''(x_0) + \frac{(2h)^3}{3!} f'''(x_0) + \cdots$$

$$f_{-2} = f(x_0-2h) = f(x_0) - 2h f'(x_0) + \frac{(2h)^2}{2!} f''(x_0) - \frac{(2h)^3}{3!} f'''(x_0) + \cdots$$

$$\vdots$$

$$f_k = f(x_0+kh) = f_0 + kh f'(x_0) + \frac{(kh)^2}{2!} f''(x_0) + \frac{(kh)^3}{3!} f'''(x_0) + \cdots \text{ where } k = 0, \pm1, \pm2, \ldots$$

Exp Derive the C.D.F of order $O(h^2)$ for $f'(x_0)$ using Taylor's Expansion.

- $f'(x_0) \approx \frac{f_1 - f_{-1}}{2} + \frac{-h^2 f'''(c)}{6}$
- Note that

$$f_1 = f_0 + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{3!} f'''(c) \text{ and}$$

$$f_{-1} = f_0 - h f'(x_0) + \frac{h^2}{2} f''(x_0) - \frac{h^3}{3!} f'''(c)$$

- Hence, $f_1 - f_{-1} = 2h f'(x_0) + \frac{2h^3}{6} f'''(c)$
- That is $\Rightarrow$

$$f'(x_0) = \frac{f_1 - f_{-1}}{2h} - \frac{h^2 f'''(c)}{6}$$

Exp Derive the C.D.F of order $O(h^2)$ for $f''(x)$ using Taylor's expansion.

$$f''(x_0) \doteq \frac{f_1 - 2f_0 + f_{-1}}{h^2} + \frac{-h^2 f^{(4)}(c)}{12}$$

- Note that

$$f_1 = f_0 + \cancel{h f'(x_0)} + \frac{h^2}{2!} f''(x_0) + \cancel{\frac{h^3}{3!} f'''(x_0)} + \frac{h^4}{4!} f^{(4)}(c)$$

$$f_{-1} = f_0 - \cancel{h f'(x_0)} + \frac{h^2}{2!} f''(x_0) - \cancel{\frac{h^3}{3!} f'''(x_0)} + \frac{h^4}{4!} f^{(4)}(c)$$

- Adding these equations $\Rightarrow$

$$f_1 + f_{-1} = 2 f_0 + h^2 f''(x_0) + \frac{h^4}{12} f^{(4)}(c)$$

- Solving for $f''(x_0)$ $\Rightarrow$

$$f''(x_0) = \frac{f_1 - 2f_0 + f_{-1}}{h^2} - \frac{h^2 f^{(4)}(c)}{12}$$

Exercise $\boxed{1}$ Derive the F.D.F of order $O(h^2)$ for $f''(x)$ using Taylor's Expansion

We need to show:

$$f''(x_0) = \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} + \frac{11}{12} h^2 f^{(4)}(c)$$

$$f_1 = f_0 + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \frac{h^4}{4!} f^{(4)}(c)$$

$$f_2 = f_0 + 2h f'(x_0) + \frac{(2h)^2}{2!} f''(x_0) + \frac{(2h)^3}{3!} f'''(x_0) + \frac{(2h)^4}{4!} f^{(4)}(c)$$

$$f_3 = f_0 + 3h f'(x_0) + \frac{(3h)^2}{2!} f''(x_0) + \frac{(3h)^3}{3!} f'''(x_0) + \frac{(3h)^4}{4!} f^{(4)}(c)$$

$$-5f_1 + 4f_2 - f_3 = \left(-5f_0 + 4f_0 - f_0\right) +$$

$$f'(x_0)\left(-5h + 8h - 3h\right) +$$

$$f''(x_0)\left(-\frac{5h^2}{2} + \frac{16h^2}{2} - \frac{9h^2}{2}\right) +$$

$$f'''(x_0)\left(\frac{-5h^3}{6} + \frac{4(8)h^3}{6} - \frac{27h^3}{6}\right) +$$

$$f^{(4)}(c)\left(\frac{-5h^4}{24} + \frac{4(16)}{24}h^4 - \frac{81h^4}{24}\right)$$

$$= -2f_0 + 0 + h^2 f''(x_0) + 0 - \frac{22}{24} h^4 f^{(4)}(c)$$

Hence, $\quad 2f_0 - 5f_1 + 4f_2 - f_3 + \frac{11}{12} h^4 f^{(4)}(c) = h^2 f''(x_0)$

$$f''(x_0) = \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} + \frac{11}{12} h^2 f^{(4)}(c)$$

Exercise Derive the B.D.F of order $o(h^2)$ for $f''(x)$ using Taylor's Expansion. $\boxed{13\,2.2}$

We need to show that:

$$f''(x_0) = \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2} + \frac{11}{12} h^2 f^{(4)}(c)$$

$$f_{-1} = f_0 - hf'(x_0) + \frac{h^2}{2!} f''(x_0) - \frac{h^3}{3!} f'''(x_0) + \frac{h^4}{4!} f^{(4)}(c)$$

$$f_{-2} = f_0 - 2hf'(x_0) + \frac{(-2h)^2}{2!} f''(x_0) + \frac{(-2h)^3}{3!} f'''(x_0) + \frac{(-2h)^4}{4!} f^{(4)}(c)$$

$$f_{-3} = f_0 - 3hf'(x_0) + \frac{(-3h)^2}{2!} f''(x_0) + \frac{(-3h)^3}{3!} f'''(x_0) + \frac{(-3h)^4}{4!} f^{(4)}(c)$$

$$\begin{aligned}
-5f_{-1} + 4f_{-2} - f_{-3} &= (-5f_0 + 4f_0 - f_0) + \\
&\quad f'(x_0)(5h - 8h + 3h) + \\
&\quad f''(x_0)\left(-\frac{5h^2}{2} + \frac{16h^2}{2} - \frac{9h^2}{2}\right) + \\
&\quad f'''(x_0)\left(\frac{5h^3}{6} - \frac{4(8)h^3}{6} + \frac{27h^3}{6}\right) + \\
&\quad f^{(4)}(c)\left(-\frac{5h^4}{24} + \frac{4(16)h^4}{24} - \frac{81h^4}{24}\right) \\
&= -2f_0 + 0 + h^2 f''(x_0) + 0 - \frac{22}{24} h^4 f^{(4)}(c)
\end{aligned}$$

Hence, $2f_0 - 5f_{-1} + 4f_{-2} - f_{-3} + \frac{11}{12} h^4 f^{(4)}(c) = h^2 f''(x_0)$

$\Rightarrow$

$$f''(x_0) = \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2} + \frac{11}{12} h^2 f^{(4)}(c)$$

Exercise Derive the C.D.F of order $o(h^4)$ for $f''(x)$ using Taylor's Expansion $\boxed{132.3}$

We need to show that

$$f''(x_0) = \frac{-f_2 + 16 f_1 - 30 f_0 + 16 f_{-1} - f_{-2}}{12 h^2} + \frac{h^4}{90} f^{(6)}(c)$$

$$f_1 = f_0 + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \frac{h^4}{4!} f^{(4)}(x_0) + \frac{h^5}{5!} f^{(5)}(x_0) + \frac{h^6}{6!} f^{(6)}(c)$$

$$f_{-1} = f_0 - h f'(x_0) + \frac{h^2}{2!} f''(x_0) - \frac{h^3}{3!} f'''(x_0) + \frac{h^4}{4!} f^{(4)}(x_0) - \frac{h^5}{5!} f^{(5)}(x_0) + \frac{h^6}{6!} f^{(6)}(c)$$

$$f_2 = f_0 + 2h f'(x_0) + \frac{(2h)^2}{2!} f''(x_0) + \frac{(2h)^3}{3!} f'''(x_0) + \frac{(2h)^4}{4!} f^{(4)}(x_0) + \frac{(2h)^5}{5!} f^{(5)}(x_0) + \frac{(2h)^6}{6!} f^{(6)}(c)$$

$$f_{-2} = f_0 - 2h f'(x_0) + \frac{(2h)^2}{2!} f''(x_0) - \frac{(2h)^3}{3!} f'''(x_0) + \frac{(2h)^4}{4!} f^{(4)}(x_0) - \frac{(2h)^5}{5!} f^{(5)}(x_0) + \frac{(2h)^6}{6!} f^{(6)}(c)$$

$$\begin{aligned}
16 f_1 + 16 f_{-1} - f_2 - f_{-2} = {} & (16 f_0 + 16 f_0 - f_0 - f_0) + \\
& f'(x_0)\left(16h - 16h - 2h + 2h\right) + \\
& f''(x_0)\left(8h^2 + 8h^2 - 2h^2 - 2h^2\right) + \\
& f'''(x_0)\left(\frac{16h^3}{6} - \frac{16h^3}{6} - \frac{8h^3}{6} + \frac{8h^3}{6}\right) + \\
& f^{(4)}(x_0)\left(\frac{16h^4}{24} + \frac{16h^4}{24} - \frac{16h^4}{24} - \frac{16h^4}{24}\right) + \\
& f^{(5)}(x_0)\left(\frac{16h^5}{5!} - \frac{16h^5}{5!} - \frac{32h^5}{5!} + \frac{32h^5}{5!}\right) + \\
& f^{(6)}(c)\left(\frac{16h^6}{720} + \frac{16h^6}{720} - \frac{64h^6}{720} - \frac{64h^6}{720}\right)
\end{aligned}$$

$$= 30 f_0 + 0 + 12h^2 f''(x_0) + 0 + 0 + 0 - \cancel{\frac{96}{720}}\,\frac{2}{15}\, h^6 f^{(6)}(c)$$

$$\Rightarrow f''(x_0) = \frac{-f_2 + 16 f_1 - 30 f_0 + 16 f_{-1} - f_{-2}}{12 h^2} + \frac{h^4}{90} f^{(6)}(c)$$

Exp Derive the B.D.F of order $o(h^2)$ with its truncation error using Newton's Polynomial.

- Recall that : $f(t) = P_n(t) + E_n(t)$ with
$f'(t) = P_n'(t) + E_n'(t)$ where

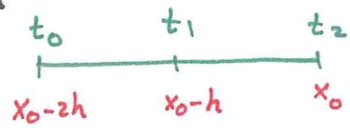
$P_n(t)$ is the Newton poly. given by

$$P_n(t) = a_0 + a_1(t-t_0) + a_2(t-t_0)(t-t_1) + \dots + a_n(t-t_0)\cdots(t-t_{n-1})$$

with $a_0 = f[t_0] = y_0$ and $a_1 = f[t_0, t_1]$, $a_2 = f[t_0, t_1, t_2]$ ...

- We need to show $f'(x_0) = \dfrac{3f_0 - 4f_{-1} + f_{-2}}{2h} + \dfrac{h^2 f'''(c)}{3}$

since $o(h^2) \Rightarrow n = 2 \Rightarrow$ find $f'(t_2)$:
(B.D.F)



$f(t) = P_2(t) + E_2(t)$

$f'(t) = P_2'(t) + E_2'(t)$ but $f'(x_0) = f'(t_2) = P_2'(t_2) + E_2'(t_2)$

- $P_2(t) = a_0 + a_1(t-t_0) + a_2(t-t_0)(t-t_1)$

$P_2'(t) = 0 + a_1 + a_2(t-t_0) + a_2(t-t_1) = a_1 + a_2[2t - t_0 - t_1]$

$P_2'(t_2) = a_1 + a_2(2t_2 - t_0 - t_1) = \boxed{a_1 + 3h\,a_2}$

- $a_1 = f[t_0, t_1] = \dfrac{f(t_1) - f(t_0)}{t_1 - t_0} = \dfrac{f_{-1} - f_{-2}}{h}$

$$a_2 = f[t_0, t_1, t_2] = \frac{f[t_1, t_2] - f[t_0, t_1]}{t_2 - t_0} = \frac{\dfrac{f[t_2] - f(t_1)}{t_2 - t_1} - \dfrac{f_{-1} - f_{-2}}{h}}{t_2 - t_0}$$

$$= \frac{\dfrac{f_0 - f_{-1}}{h} - \dfrac{f_{-1} - f_{-2}}{h}}{2h} = \frac{f_0 - 2f_{-1} + f_{-2}}{2h^2}$$

- Hence, $P_2'(t_2) = a_1 + 3h\,a_2 = \dfrac{f_{-1} - f_{-2}}{h} + 3\cancel{h}\left(\dfrac{f_0 - 2f_{-1} + f_{-2}}{2h^{\cancel{2}}}\right)$

$$= \frac{3f_0 - 4f_{-1} + f_{-2}}{2h}$$

• To find the error $E_2'(t_2)$ $\Rightarrow$

$\Rightarrow$ Recall the error term $E_2(t) = \dfrac{f'''(c)\,(t-t_0)(t-t_1)(t-t_2)}{3!}$

$\Rightarrow$ Now $E_2'(t) = \dfrac{f'''(c)}{6}\Big[(t-t_0)(t-t_1) + (t-t_2)\big((t-t_0) + (t-t_1)\big)\Big]$

$$E_2'(t_2) = \frac{f'''(c)}{6}\Big[(t_2-t_0)(t_2-t_1)\Big]$$

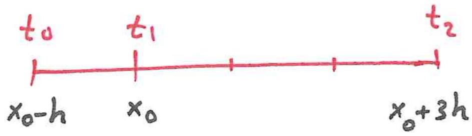
$$= \frac{f'''(c)\,(2h)(h)}{6} = \frac{h^2 f'''(c)}{3}$$

Exp Derive the D.F $f''(x_0) = \dfrac{f_3 - 4f_0 + 3f_{-1}}{6h^2} - \dfrac{2h f'''(c)}{3}$

using Lagrange polynomial.

• $n=2 \Rightarrow$ $f(t) = P_2(t) + E_2(t)$

$$f''(t) = P_2''(t) + E_2''(t)$$



• $f''(x_0) = f''(t_1) = P_2''(t_1) + E_2''(t_1)$ where $P_2(t)$ is Lagrange's poly. given by

$$P_2(t) = y_0 \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)} + y_1 \frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)} + y_2 \frac{(t-t_0)(t-t_1)}{(t_2-t_0)(t_2-t_1)}$$

$$= f_{-1} \frac{(t-t_1)(t-t_2)}{(-h)(-4h)} + f_0 \frac{(t-t_0)(t-t_2)}{(h)(-3h)} + f_3 \frac{(t-t_0)(t-t_1)}{(4h)(3h)}$$

$$P_2'(t) = \frac{f_{-1}}{4h^2}(t-t_1+t-t_2) - \frac{f_0}{3h^2}(t-t_0+t-t_2) + \frac{f_3}{12h^2}(t-t_0+t-t_1)$$

$$P_2''(t) = \frac{2f_{-1}}{4h^2} - \frac{2f_0}{3h^2} + \frac{2f_3}{12h^2}$$

$$= \frac{2f_3 - 8f_0 + 6f_{-1}}{12h^2}$$

$$= \frac{f_3 - 4f_0 + 3f_{-1}}{6h^2} \quad ✓$$

• To find the error $E_2''(t_1) \Rightarrow$

$\Rightarrow$ Recall the error term $E_2(t) = \dfrac{f'''(c)(t-t_0)(t-t_1)(t-t_2)}{3!}$

$$\Rightarrow E_2'(t) = \frac{f'''(c)}{6}\Big[(t-t_0)(t-t_1) + (t-t_2)\big((t-t_0)+(t-t_1)\big)\Big]$$

$$E_2''(t) = \frac{f'''(c)}{6}\Big[(t-t_0) + (t-t_1) + (t-t_2) + (t-t_0) + (t-t_1) + (t-t_2)\Big]$$

$$= \frac{f'''(c)}{3}\Big[(t-t_0) + (t-t_1) + (t-t_2)\Big]$$

$$\Rightarrow E_2''(t_1) = \frac{f'''(c)}{3}\Big[(t_1-t_0) + 0 + (t_1-t_2)\Big]$$

$$= \frac{f'''(c)}{3}\Big[h - 3h\Big]$$

$$= \frac{-2h\,f'''(c)}{3}$$