Error Analysis and Optimal step Size

- In any D.F. $\Rightarrow$

Total Error = Round off Error + Truncation Error

$$\boxed{E_{Total}(f,h) = E_{round}(f,h) + E_{trun}(f,h)}$$

- The Round off Error $E_{round}(f,h)$ in any D.F :

$$f_k = y_k + e_k \quad , \; k = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow f_0 = y_0 + e_0 \; , \; f_1 = y_1 + e_1 \; , \; f_2 = y_2 + e_2$$
$$f_{-1} = y_{-1} + e_{-1} \; , \; f_{-2} = y_{-2} + e_{-2} \; \ldots$$

- **Remark**: The magnitude of the round off error is

$$|e_k| \leq \epsilon = 5 \times 10^{-10}$$

- The truncation error $E_{trun}(f,h)$ depends on the form of the D.F.

- To find the optimal step size h for a given D.F, we differentiate $E_{Total}(f,h)$ w.r.t h and find the critical value.

*Exp • Let $f(x) = \sin x$

• Estimate $f'(1)$ using the C.D.F of order $O(h^2)$ with $h = 0.01$ and $h = 0.001$ and $h = 0.0001$ and compare with the true value.

• True Value : $f'(x) = \cos x \Rightarrow f'(1) = \boxed{0.5403023059}$

• $f'(x_0) \simeq \dfrac{f(x_0+h) - f(x_0-h)}{2h}$

• $h = 0.01 \Rightarrow f'(1) = \dfrac{f(1.01) - f(0.99)}{2(0.01)} = 0.5402933009$

• $h = 0.001 \Rightarrow f'(1) = \dfrac{f(1.001) - f(0.999)}{2(0.001)} = 0.5403022158$

• $\boxed{h = 0.0001} \Rightarrow f'(1) = \dfrac{f(1.0001) - f(0.9999)}{2(0.0001)} = \boxed{0.540302305}$

optimal since it gives zero error

Exp Find the optimal step size h for the C.D.F of order $O(h^2)$ using to estimate $f'(x_0)$.

• $f'(x_0) = \dfrac{f(x_0+h) - f(x_0-h)}{2h} + E_{trun}(f,h)$

$= \dfrac{f_1 - f_{-1}}{2h} - \dfrac{h^2 f'''(c)}{6}$

$= \dfrac{y_1 + e_1 - (y_{-1} + e_{-1})}{2h} - \dfrac{h^2 f'''(c)}{6}$

$= \dfrac{y_1 - y_{-1}}{2h} + \underbrace{\dfrac{e_1 - e_{-1}}{2h}}_{\text{Roundoff Error}} + \underbrace{\dfrac{-h^2 f'''(c)}{6}}_{\text{Truncation Error}}$

- Hence, the total error is

$$E_{total}(f,h) = E_{round\,off}(f,h) + E_{trun}(f,h)$$

$$= \frac{e_1 - e_{-1}}{2h} + \frac{-h^2 f'''(c)}{6}$$

- Now $$|E_{total}| \leq \left|\frac{e_1 - e_{-1}}{2h}\right| + \left|\frac{h^2 f'''(c)}{6}\right|$$

$$\leq \left|\frac{e_1}{2h}\right| + \left|\frac{e_1}{2h}\right| + \frac{h^2 M}{6}$$

$$\leq \frac{\epsilon}{2h} + \frac{\epsilon}{2h} + \frac{h^2 M}{6}$$ by Remark page 136

$$= \frac{\epsilon}{h} + \frac{h^2 M}{6}$$

$$= \phi(h)$$

- Now set $\phi'(h) = 0$ and find the critical value h^*

$$-\frac{\epsilon}{h^2} + \frac{hM}{3} = 0 \iff h^3 M = 3\epsilon$$

$$\iff h^* = \left(\frac{3\epsilon}{M}\right)^{\frac{1}{3}}$$

- Note that in *Exp $\Rightarrow M = \max|f'''(c)| = 1$ since $f(x) = \sin x$

$$\Rightarrow h^* = (3\epsilon)^{\frac{1}{3}} = (3 \times 5 \times 10^{-10})^{\frac{1}{3}} \simeq 0.001145$$

$$\Rightarrow h^* \approx 0.0001$$ ✓

- Remember that for C.D.F of order $o(h^2)$ to estimate $f'(x_0) \Rightarrow$ we have $f'(x_0) = \frac{f_1 - f_{-1}}{2h} - \frac{h^2 f'''(c)}{6}$ with

$$\phi(h) = \frac{\epsilon}{h} + \frac{h^2 M}{6} \quad \text{and} \quad h^* = \left(\frac{3\epsilon}{M}\right)^{\frac{1}{3}}$$

Exp Find the optimal step size h for the C.D.F of order $O(h^2)$ in estimating $f''(x_0)$.

- $$f''(x_0) = \frac{f_1 - 2f_0 + f_{-1}}{h^2} + E_{trun}(f,h)$$

$$= \frac{y_1 + e_1 - 2(y_0 + e_0) + y_{-1} + e_{-1}}{h^2} + \frac{-h^2 f^{(4)}(c)}{12}$$

$$= \frac{y_1 - 2y_0 + y_{-1}}{h^2} + \underbrace{\frac{e_1 - 2e_0 + e_{-1}}{h^2}}_{\text{Round off Error}} + \underbrace{\frac{-h^2 f^{(4)}(c)}{12}}_{\text{Truncation Error}}$$

- Hence, $E_{total}(f,h) = E_{roundoff}(f,h) + E_{trun}(f,h)$

$$= \frac{e_1 - 2e_0 + e_{-1}}{h^2} + \frac{-h^2 f^{(4)}(c)}{12}$$

- Now $$|E_{total}| \le \left|\frac{e_1 - 2e_0 + e_{-1}}{h^2}\right| + \left|\frac{h^2 f^{(4)}(c)}{12}\right|$$

$$\le \frac{\epsilon + 2\epsilon + \epsilon}{h^2} + \frac{h^2 M}{12}$$

$$= \frac{4\epsilon}{h^2} + \frac{h^2 M}{12}$$

$$= \phi(h)$$

- $$\phi'(h) = 0 \iff \frac{-8\epsilon}{h^3} + \frac{hM}{6} = 0 \iff$$

$$h^* = \left(\frac{48\epsilon}{M}\right)^{\frac{1}{4}}$$

Exp • Let $f(x) = \ln x$ where $0.1 \leq x \leq 0.5$

• Find the best step size h for the C.D.F of order $O(h^2)$ in estimating $f''(x_0)$

• Using the previous Exp $\Rightarrow h^* = \left(\frac{48\epsilon}{M}\right)^{\frac{1}{4}}$

• To find $M \Rightarrow f'(x) = \frac{1}{x} \Rightarrow f''(x) = \frac{-1}{x^2}$

$$\Rightarrow f'''(x) = \frac{2}{x^3} \Rightarrow f^{(4)}(x) = \frac{-6}{x^4}$$

$$M = \underset{0.1 \leq x \leq 0.5}{\text{Max}} \left| f^{(4)}(x) \right| = \underset{0.1 \leq x \leq 0.5}{\text{Max}} \frac{6}{x^4} = \frac{6}{(0.1)^4} = 60000$$

• $$h^* = \left(\frac{48 \times 5 \times 10^{-10}}{60000}\right)^{\frac{1}{4}} = \left(4 \times 10^{-13}\right)^{\frac{1}{4}} = 0.0007952707$$

Exp Find the optimal h for $f(x) = e^{-x}$, $1 \leq x \leq 2$ if the C.D.F of order $O(h^4)$ is used to estimate $f'(x_0)$.

• $$f'(x_0) = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

• Similarly to what have been done befor, we could arrive:

$$\phi(h) = \frac{18\epsilon}{12h} + \frac{h^4 M}{30} = \frac{3\epsilon}{2h} + \frac{h^4 M}{30}$$

• $$\phi'(h) = \frac{-3\epsilon}{2h^2} + \frac{2h^3 M}{15} = 0 \Leftrightarrow h^* = \left(\frac{45\epsilon}{4M}\right)^{\frac{1}{5}}$$

• To find $M \Rightarrow f' = -e^{-x} = f'' = f^{(5)} \Rightarrow M = \underset{1 \leq x \leq 2}{\text{Max}} \left| f^{(5)}(x) \right| = \left. \frac{1}{e^x} \right|_{x=1}$

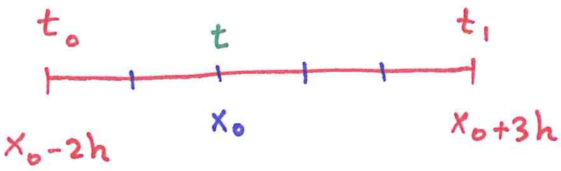
$$= \frac{1}{e} = 0.3678794412$$

• Hence, $$h^* = \left(\frac{45 \times 5 \times 10^{-10}}{0.3678794412}\right)^{\frac{1}{5}}$$

$$= 0.0360823986$$

Exp Use the points: $x_0 - 2h$, $x_0 + 3h$ to estimate $f'(x)$ with its truncation error using Newton's Interpolation.

- $n = 1 \Rightarrow$ Newton's Poly. is $P_1(t) = a_0 + a_1(t - t_1)$



$$P_1'(t) = a_1 = f[t_0, t_1]$$

$$= \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

$$= \frac{f_3 - f_{-2}}{5h}$$

$$= P_1'(x_0)$$

- $f(t) = P_1(t) + E_1(t)$

$$f'(t) = P_1'(t) + E_1'(t)$$

$$f'(x_0) = P_1'(x_0) + E_1'(x_0)$$

- Note that $E_1(t) = \dfrac{f''(c)(t - t_0)(t - t_1)}{2}$

$$E_1'(t) = \frac{f''(c)}{2}\left[(t - t_0) + (t - t_1)\right]$$

$$E_1'(x_0) = \frac{f''(c)}{2}\left[2h + (-3h)\right]$$

$$= -\frac{h f''(c)}{2}$$

- Hence, $f'(x_0) = P_1'(x_0) + E_1'(x_0)$

$$= \frac{f_3 - f_{-2}}{5h} - \frac{h f''(c)}{2}$$
