

Ch 7

Numerical Integration

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* How can we estimate $\int_a^b f(x) dx$?

We use quadrature formula $Q[f]$.

Def (Quadrature Formula)

- Suppose that $a = x_0 < x_1 < \dots < x_m = b$.
- A formula of the form $Q[f] = \sum_{k=0}^m w_k f(x_k)$
 $= w_0 f(x_0) + w_1 f(x_1) + \dots + w_m f(x_m)$

with the property that

$$\int_a^b f(x) dx = Q[f] + E[f]$$

is called a numerical integration (or quadrature formula).

- $E[f]$ is called the truncation error for integration.
- x_0, x_1, \dots, x_m are called the quadrature nodes.
- w_0, w_1, \dots, w_m are called the weights.

* We will study two types of $Q[f]$:

[1] Closed Newton-Cotes Quadrature Formula:

- (a) Trapezoidal Rule
- (b) Simpson's Rule
- (c) Simpson's $\frac{3}{8}$ Rule

[2] Gauss-Legendre Formula:

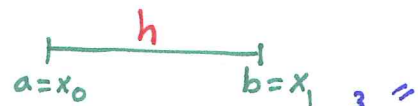
- (a) $G_1(f)$
- (b) $G_2(f)$
- (c) $G_3(f)$

Closed Newton - Coles Quadrature Formula

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1h • Assume that $x_k = x_0 + kh$ are equally spaced nodes with $f_k = f(x_k)$

• Then [a] Trapezoidal Rule is



$$\int_a^b f(x) dx = \int_{x_0}^{x_1} f(x) dx \approx \frac{h}{2} (f_0 + f_1) \text{ with error } \frac{-h^3 f^{(3)}(c)}{12}$$

$\underbrace{\hspace{10em}}_{Q[f]} \qquad \underbrace{\hspace{10em}}_{E[f]}$

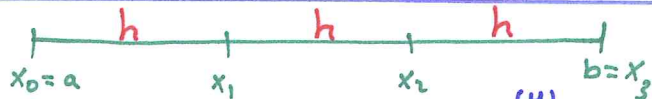
[b] Simpson's Rule is



$$\int_a^b f(x) dx = \int_{x_0}^{x_2} f(x) dx \approx \frac{h}{3} (f_0 + 4f_1 + f_2) \text{ with error } \frac{-h^5 f^{(4)}(c)}{90}$$

$\underbrace{\hspace{10em}}_{Q[f]} \qquad \underbrace{\hspace{10em}}_{E[f]}$

[c] Simpson's $3/8$ Rule is



$$\int_a^b f(x) dx = \int_{x_0}^{x_3} f(x) dx \approx \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3) \text{ with error } \frac{-3h^5 f^{(4)}(c)}{80}$$

$\underbrace{\hspace{10em}}_{Q[f]} \qquad \underbrace{\hspace{10em}}_{E[f]}$

Exp Estimate $\int_0^1 (1 + e^{-x} \sin(4x)) dx$ using

[1] Trapezoidal Rule

$$\int_0^1 (1 + e^{-x} \sin(4x)) dx \approx \frac{h}{2} (f_0 + f_1)$$

$$= \frac{1}{2} (1 + 0.72159)$$

$$= 0.86079$$



$$f_0 = f(0) = 1$$

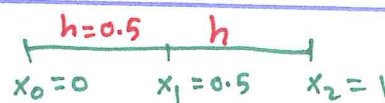
$$f_1 = f(1) = 0.72159$$

[2] Simpson's Rule

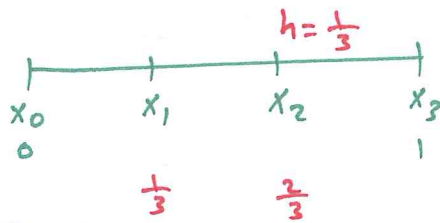
$$\int_0^1 (1 + e^{-x} \sin(4x)) dx \approx \frac{0.5}{3} (f(0) + 4f(0.5) + f(1))$$

$$= \frac{1}{6} (1 + 4(1.55152) + 0.72159)$$

$$= 1.32128$$



3 Simpson's $\frac{3}{8}$ Rule



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$$\int_0^1 (1 + e^x \sin(4x)) dx \approx \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3)$$

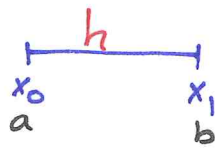
$$= \frac{3(\frac{1}{3})}{8} [1 + 3(1.69642) + 3(1.23447) + 0.72159]$$

$$= 1.31440$$

Exp Derive the Trapezoidal Rule $\int_a^b f(x) \approx \frac{h}{2} (f_0 + f_1)$

• Since $a = x_0$ and $b = x_1 \Rightarrow n = 1$

• Use Newton Interpolation $\Rightarrow f(x) \approx P_1(x)$



$$P_1(x) = a_0 + a_1(x - x_0)$$

• Hence, $\int_a^b f(x) dx \approx \int_a^b P_1(x) dx = \int_{x_0}^{x_1} (a_0 + a_1(x - x_0)) dx$

$$= a_0(x_1 - x_0) + a_1 \left. \frac{(x - x_0)^2}{2} \right|_{x_0}^{x_1}$$

$$= a_0 h + \frac{a_1 h^2}{2}$$

or change of variables

$$x - x_0 = ht$$

$$dx = h dt$$

$$x = x_0 \Rightarrow t = 0$$

$$x = x_1 \Rightarrow t = 1$$

$$\int_{x_0}^{x_1} (a_0 + a_1(x - x_0)) dx =$$

$$\int_0^1 (a_0 + a_1 ht) h dt$$

$$= ha_0 + \frac{a_1 h^2}{2}$$

• But $a_0 = f[x_0] = f(x_0) = f_0$

$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f_1 - f_0}{h}$$

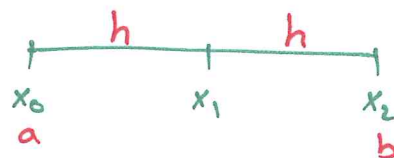
• Thus, $\int_a^b f(x) dx \approx f_0 h + \frac{f_1 - f_0}{2h} h^2$

$$= \frac{h}{2} (f_0 + f_1)$$

EXP Derive the Simpson's Rule $\int_a^b f(x) dx \approx \frac{h}{3} (f_0 + 4f_1 + f_2)$ 145

• Here $n = 2$ since we have 3 points

• Use Lagrange Poly. $\Rightarrow f(x) \approx P_2(x)$



where $P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

• Hence, $\int_a^b f(x) dx = \int_{x_0}^{x_2} P_2(x) dx$

$$= \int_{x_0}^{x_2} \frac{f_0}{2h^2} (x-x_1)(x-x_2) dx + \int_{x_0}^{x_2} \frac{f_1}{h^2} (x-x_0)(x-x_2) dx + \int_{x_0}^{x_2} \frac{f_2}{2h^2} (x-x_0)(x-x_1) dx$$

• Use the following change of variables:

$$\left. \begin{array}{l} x - x_0 = ht \\ dx = h dt \end{array} \right\} \Rightarrow \begin{array}{l} \text{when } x = x_0 \Rightarrow t = 0 \\ x = x_2 \Rightarrow t = 2 \end{array}$$

• Note that $x - x_1 = x - (x_0 + h) = (x - x_0) - h = ht - h = h(t-1)$

$$x - x_2 = x - (x_0 + 2h) = (x - x_0) - 2h = ht - 2h = h(t-2)$$

• Hence, $\int_{x_0}^{x_2} P_2(x) dx = \int_0^2 \frac{f_0}{2h^2} h(t-1)h(t-2) h dt - \int_0^2 \frac{f_1}{h^2} h(t)h(t-2) h dt + \int_0^2 \frac{f_2}{2h^2} h(t)h(t-1) h dt$

$$= \frac{hf_0}{2} \int_0^2 (t^2 - 3t + 2) dt - hf_1 \int_0^2 (t^2 - 2t) dt + \frac{hf_2}{2} \int_0^2 (t^2 - t) dt$$

$$= \frac{hf_0}{2} \left(\frac{t^3}{3} - \frac{3}{2}t^2 + 2t \right) \Big|_0^2 - hf_1 \left(\frac{t^3}{3} - t^2 \right) \Big|_0^2 + \frac{hf_2}{2} \left(\frac{t^3}{3} - \frac{t^2}{2} \right) \Big|_0^2$$

$$= \frac{hf_0}{2} \left(\frac{8}{3} \right) - hf_1 \left(\frac{-4}{3} \right) + \frac{hf_2}{2} \left(\frac{8}{3} \right)$$

$$= \frac{h}{3} [f_0 + 4f_1 + f_2]$$