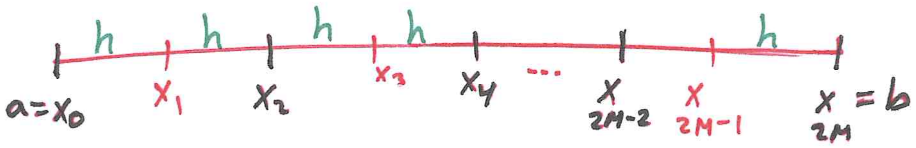
Composit Simpson Rule (CSR)

This method approximates the area under the curve $y = f(x)$ over $[a,b]$.

Th (CSR)

- Assume that the interval $[a,b]$ is subdivided into $2M$ subintervals $[x_k, x_{k+1}]$ each of width $h = \frac{b-a}{2M}$ using equally spaced nodes $x_k = a + kh$ for $k = 0,1,2,\dots,2M$:



- Then, the Composite Simpson Rule is

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_{2M}} f(x)\,dx \approx S(f,h) = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots + \int_{x_{2M-2}}^{x_{2M}} f(x)\,dx$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right] + \dots + \frac{h}{3}\left[f_{2M-2} + 4f_{2M-1} + f_{2M}\right]$$

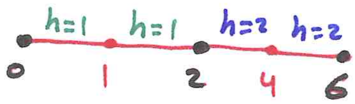
- Furthermore, the total error of $S(f,h)$ is

$$E_S(f,h) = \frac{-h^5 f^{(4)}(c)}{90} \cdot M = \frac{-h^4 f^{(4)}(c)}{180}(b-a)$$

Exp Given

x	0	1	2	4	6
$f(x)$	2	-1	3	0	10

Estimate $\int_0^6 f(x)\,dx$ using CSR.



$$\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx$$

$$= \frac{1}{3}\left[f(0) + 4f(1) + f(2)\right] + \frac{2}{3}\left[f(2) + 4f(4) + f(6)\right]$$

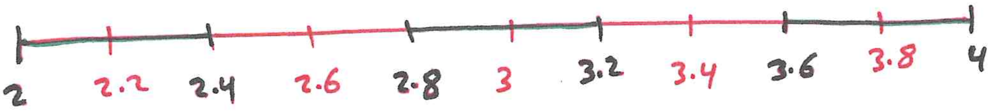
$$= \frac{1}{3}\left[2 - 4 + 3\right] + \frac{2}{3}\left[3 + 0 + 10\right]$$

$$= \frac{1}{3} + \frac{26}{3} = \frac{27}{3} = 9$$

Exp Use CSR to estimate $\int_2^4 e^x\,dx$ with 5 compositions. $\boxed{157}$

Use 4 chopping digits.

- $h = \dfrac{b-a}{2M} = \dfrac{4-2}{2(5)} = \dfrac{2}{10} = 0.2$ and $f(x) = e^x = (2.718)^x$



- $$\int_2^4 e^x\,dx = \int_2^{2.4} e^x\,dx + \int_{2.4}^{2.8} e^x\,dx + \int_{2.8}^{3.2} e^x\,dx + \int_{3.2}^{3.6} e^x\,dx + \int_{3.6}^{4} e^x\,dx$$

$$= \frac{0.2}{3}\left[f(2) + 4f(2.2) + f(2.4)\right] + \frac{0.2}{3}\left[f(2.4) + 4f(2.6) + f(2.8)\right]$$

$$+ \frac{0.2}{3}\left[f(2.8) + 4f(3) + f(3.2)\right] + \frac{0.2}{3}\left[f(3.2) + 4f(3.4) + f(3.6)\right]$$

$$+ \frac{0.2}{3}\left[f(3.6) + 4f(3.8) + f(4)\right]$$

$$= 0.06666\left[(2.718)^2 + 4(2.718)^{2.2} + 2(2.718)^{2.4} + 4(2.718)^{2.6} + 2(2.718)^{2.8} + 4(2.718)^3 + 2(2.718)^{3.2} + 4(2.718)^{3.4} + 2(2.718)^{3.6} + 4(2.718)^{3.8} + (2.718)^4\right]$$

$$= 0.06666\left[7.387 + 36.08 + 22.04 + 53.84 + 32.86 + 80.28 + 49.04 + 119.8 + 73.16 + 178.7 + 54.57\right]$$

$$= 0.06666(707.4)$$

$$= 47.15$$

Note that the True Value is $\int_2^4 e^x\,dx = e^x\Big|_2^4 = e^4 - e^2 = 47.2090939342$

Exp Find the number of compositions and the step size needed to estimate $\int_2^7 \frac{dx}{x}$ with accuracy 5×10^{-9} using

1. CTR
2. CSR

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1. $|E| = \left| \frac{-h^2 f''(c)}{12}(b-a) \right| \le 5\times10^{-9}$

$a=2$, $b=7$

$f(x) = \frac{1}{x}$

$f' = -\frac{1}{x^2}$

$f'' = \frac{2}{x^3} \le \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$

$$\left| \frac{h^2 (\frac{1}{4})}{12} (\cancel{7-2}) \right| \le \cancel{5}\times10^{-9}$$

$$h \le \sqrt{12\times10^{-9}\times4} = 0.000219089$$

$$M = \frac{b-a}{h} \ge \frac{5}{0.000219089} = 22821.77562543$$

so the number of compositions is $M \ge 22822$ and # of points $= M+1$

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2. $|E| = \left| \frac{h^4 f^{(4)}(c)}{180}(b-a) \right| \le 5\times10^{-9}$

$f'''(x) = -\frac{6}{x^4}$

$f^{(4)} = \frac{24}{x^5} \le \frac{24}{2^5} = \frac{24}{32} = \frac{3}{4}$

$$\frac{h^4 (\frac{3}{4})(\cancel{5})}{\cancel{180}_{60}} \le \cancel{5}\times10^{-9}$$

$$h \le (240\times10^{-9})^{\frac{1}{4}} = 0.0221336384$$

$$M = \frac{b-a}{2h} \ge \frac{5}{0.0442672768} = 112.9502504206$$

Hence, $M \ge 113$