Gauss - Legendre Translation

- How to use $G_2(f)$ and $G_3(f)$ to estimate $\int_a^b f(x)\,dx$
- we use change of variables to transform the limits of integration from $[a,b]$ to $[-1,1]$:

$$x = \frac{a+b}{2} + \frac{b-a}{2}t \quad \Rightarrow \quad dx = \frac{b-a}{2}dt$$

- when $t=-1 \Rightarrow x = \frac{a+b}{2} - \frac{b-a}{2} = a$

  $t = 1 \Rightarrow x = \frac{a+b}{2} + \frac{b-a}{2} = b$

- Hence, $\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}t\right)\frac{b-a}{2}\,dt$

$$= \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}t\right)dt$$

$\Downarrow$

$$G_2(f) = \frac{b-a}{2}\left[f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(\frac{-1}{\sqrt{3}}\right)\right) + f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right)\right)\right]$$

and

$$G_3(f) = \frac{b-a}{2}\left[\frac{5f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(-\sqrt{\frac{3}{5}}\right)\right) + 8f\left(\frac{a+b}{2}\right) + 5f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(\sqrt{\frac{3}{5}}\right)\right)}{9}\right]$$

Exp Use two-points Gauss-Legendre rule to approximate

$$\int_1^5 e^{-x}\,dx = -e^{-x}\Big|_1^5 = -e^{-5} + e^{-1} = 0.3746173882$$

- $x = \frac{a+b}{2} + \frac{b-a}{2}t = 3+2t$

  $dx = 2\,dt$

- $$\int_1^5 e^{-x}\,dx = \int_{-1}^{1} e^{-(3+2t)}\,2\,dt = 2\left[f\left(\tfrac{-1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}\right)\right]$$

$$= 2\left[e^{-(3+\frac{-2}{\sqrt{3}})} + e^{-(3+\frac{2}{\sqrt{3}})}\right] = 0.3473369892$$

Exp Use three-points Gauss-Legendre rule to estimate

$$\int_1^5 \frac{dx}{x} = \ln x\Big|_1^5 = \ln 5 = 1.6094379124.$$ Use 4 chopping digits.

- $x = \frac{a+b}{2} + \frac{b-a}{2}t = 3+2t$ with $dx = 2\,dt$

- Now $\Rightarrow$ $\displaystyle\int_1^5 \frac{dx}{x} = \int_{-1}^{1} \frac{2\,dt}{3+2t}$ with $f(t) = \frac{2}{3+2t}$

$$= \frac{5f(-\sqrt{\tfrac{3}{5}}) + 8f(0) + 5f(\sqrt{\tfrac{3}{5}})}{9}$$

$$= \frac{1}{9}\left[5f(-0.7745) + 8f(0) + 5f(0.7745)\right]$$

$$= 0.1111\left[5(1.378) + 8(0.6666) + 5(0.4396)\right]$$

$$= 0.1111\left[6.890 + 5.332 + 2.198\right]$$

$$= 0.1111(14.41)$$

$$= 1.6$$