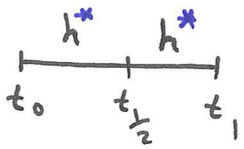
4 Runge-Kutta Method of order 4 (RK4)

- Given an IVP: $y' = f(t,y)\ ,\ y(t_0) = y_0$

$$\int_{t_0}^{t_1} y'(t)\,dt = \int_{t_0}^{t_1} f(t,y)\,dt \quad \cdots \ *$$

- Apply Simpson's Rule $\int_{t_0}^{t_1} f(t)\,dt \approx \frac{h^*}{3}\left(f_0 + 4 f_{\frac{1}{2}} + f_1\right)$



with $h^* = \frac{h}{2} \Rightarrow *$ becomes

$$\$ \ \cdots \quad y(t_1) - y(t_0) = \frac{h}{6}\left[f(t_0,y_0) + 4 f\left(t_{\frac{1}{2}}, y_{\frac{1}{2}}\right) + f(t_1,y_1)\right]$$

where $t_{\frac{1}{2}}$ is the midpoint of the interval,

- Take $f_1 = f(t_0, y_0)$,

$f_4 = f(t_1, y_1)$,

$\frac{f_2 + f_3}{2} \approx f\left(t_{\frac{1}{2}}, y_{\frac{1}{2}}\right)$ " average of f_2 and f_3 "

and substitute them in \$ $\Rightarrow$

$$y_1 = y_0 + \frac{h}{6}\left[f_1 + 2(f_2 + f_3) + f_4\right]$$

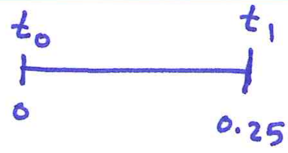
In general: $y_{k+1} = y_k + \frac{h}{6}\left[f_1 + 2f_2 + 2f_3 + f_4\right]$ where

$$f_1 = f(t_k, y_k)$$
$$f_2 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_1\right)$$
$$f_3 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_2\right)$$
$$f_4 = f\left(t_k + h, y_k + h f_3\right)$$

$k = 0, 1, 2, \ldots, n-1$

Exp Given the IVP: $y' = \dfrac{t-y}{2}$, $y(0) = 1$
Estimate $y(0.25)$ using RK4 with step size $h = \frac{1}{4}$
and use 4 chopping digits

- $t_0 = 0$, $y_0 = 1$, $h = 0.25$ and



$$f(t, y) = \frac{t-y}{2}$$

- $f_1 = f(t_0, y_0) = f(0, 1) = \dfrac{0-1}{2} = -0.5$

- $f_2 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_1\right) = f\left(0 + \frac{0.25}{2}, 1 + \frac{0.25}{2}(-0.5)\right)$

$$= f(0.125, 0.9375) = \frac{0.125 - 0.9375}{2} = -0.4062$$

- $f_3 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_2\right) = f\left(0.125, 1 + 0.125(-0.4062)\right)$

$$= f(0.125, 0.9492) = \frac{0.125 - 0.9492}{2} = -0.4121$$

- $f_4 = f(t_0 + h, y_0 + h f_3) = f\left(0 + 0.25, 1 + 0.25(-0.4121)\right)$

$$= f(0.25, 0.897) = \frac{0.25 - 0.897}{2} = -0.3235$$

- Hence, $y_1 = y_0 + \dfrac{h}{6}\left[f_1 + 2f_2 + 2f_3 + f_4\right]$

$$= 1 + \frac{0.25}{6}\left[-0.5 - 2(0.4062) - 2(0.4121) - 0.3235\right]$$

$$= 1 + 0.04166\left[-0.5 - 0.8124 - 0.8242 - 0.3235\right]$$

$$= 1 - 0.1024$$

$$= 0.8976$$