1.3 Error Analysis

Def Assume $\tilde{p}$ is approximation to p, where $p \neq 0$. Then,

- The (absolute) error is $E_p = |p - \tilde{p}|$ and
- the relative error is $R_p = \frac{|p-\hat{p}|}{|p|}$ which expresses the error as percentage of the true value.

Exp Find the error and relative error for the following cases:

1. $x = 3.141592$ and $\tilde{x} = 3.14$

$$\text{Error} = E_x = |x - \tilde{x}| = |3.141592 - 3.14| = 0.001592$$

$$\text{Relative Error} = R_x = \frac{|x-\tilde{x}|}{|x|} = \frac{0.001592}{3.141592} = 0.000507$$

$\tilde{x}$ is good approx. of x

2. $y = 1\,000\,000$ and $\hat{y} = 999\,996$

$$E_y = |y - \hat{y}| = |1\,000\,000 - 999\,996| = 4$$

$$R_y = \frac{|y-\hat{y}|}{|y|} = \frac{4}{1\,000\,000} = 4 \times 10^{-6} = 0.000004$$

$\hat{y}$ is good approx. of y

3. $z = 0.000\,012$ and $\tilde{z} = 0.000\,009$

$$E_z = |z - \hat{z}| = |0.000\,012 - 0.000\,009| = 0.000\,003$$

$$R_z = \frac{|z-\hat{z}|}{|z|} = \frac{0.000\,003}{0.000\,012} = 0.25$$

$\tilde{z}$ is bad approx. of z

Remarks

1. $\tilde{x}$ is a good estimate for x since there is no much difference between E_x and R_x and so any of them could be used to determine the accuracy of $\tilde{x}$.
2. $\tilde{y}$ is good estimate for y since R_y is small (even if E_y is large since y is of magnitude 10^6)
3. $\tilde{z}$ is bad approximation for z even that E_z is the smallest of the three cases. This because R_z is the largest.

Exp

0.000004321	4 significant digits
3.10045	6 significant digits
$2 \times 10^{-4} = 0.0002$	1 significant digits

Def The number $\tilde{p}$ approximates p to d significant digits if d is the largest non-negative integer s.t

$$R_p < 5 \times 10^{-d}$$

Exp $\boxed{1}$ $x = 3.141592$ and $\tilde{x} = \underline{3.14}$

$$R_x = 0.000507 < 0.005 = 5 \times 10^{-3} \Leftrightarrow d = 3$$

$\boxed{2}$ $y = 1000\,000$ and $\tilde{y} = \underline{999\,996}$

$$R_y = 0.000\,004 < 0.000\,005 = 5 \times 10^{-6} \Leftrightarrow d = 6$$

$\boxed{3}$ $z = 0.000\,012$ and $\bar{z} = 0.000\,00\underline{9}$

$$R_z = 0.25 < 0.5 = 5 \times 10^{-1} \Leftrightarrow d = 1$$

انتشار الخطأ **Propagation of Error**

- Assume the number p is approximated by $\tilde{p}$ with error ϵ_p $(p = \tilde{p} + \epsilon_p)$
- " " " q " " " $\tilde{q}$ " " ϵ_q $(q = \tilde{q} + \epsilon_q)$

Exp [1] Describe the error in their sum

$$p + q = (\tilde{p} + \epsilon_p) + (\tilde{q} + \epsilon_q)$$
$$= (\tilde{p} + \tilde{q}) + (\epsilon_p + \epsilon_q)$$

Hence, the error of the sum is the sum of the errors.

Exp [2] Assume $p \neq 0$ and $q \neq 0$.
Assume $\tilde{p}$ and $\tilde{q}$ are good approximations for p and q.
Show that the relative error in the product pq is approximately the sum of the relative errors in the approximation $\tilde{p}$ and $\tilde{q}$.

- $pq = (\tilde{p} + \epsilon_p)(\tilde{q} + \epsilon_q)$

$$= \tilde{p}\tilde{q} + \tilde{p}\epsilon_q + \tilde{q}\epsilon_p + \epsilon_p\epsilon_q$$

$$pq - \tilde{p}\tilde{q} = \tilde{p}\epsilon_q + \tilde{q}\epsilon_p + \epsilon_p\epsilon_q$$

$p - \tilde{p} \approx 0$

- Since $p \neq 0$ and $q \neq 0 \Rightarrow$ their relative errors

$q - \tilde{q} \approx 0$

$$R_{pq} = \frac{pq - \tilde{p}\tilde{q}}{pq} = \frac{\tilde{p}\epsilon_q}{pq} + \frac{\tilde{q}\epsilon_p}{qp} + \frac{\epsilon_p\epsilon_q}{pq}$$

$$\approx \frac{\epsilon_q}{q} + \frac{\epsilon_p}{p} + 0$$

$$= R_q + R_p$$

- This is because $\tilde{p}$ and $\tilde{q}$ are good approximation for p and q

$\Rightarrow \frac{\tilde{p}}{p} \approx 1$ and $\frac{\tilde{q}}{q} \approx 1$

Normalized decimal form

Any real number p can be written in normalized decimal form:

$$p = \pm\, 0.d_1 d_2 d_3 \cdots d_k\, d_{k+1} \cdots \times 10^n$$

where $d_1 \neq 0$ and $d_j \in \{0,1,2,\ldots,9\}$ for $j>1$.

Exp
- $0.01234 = 0.1234 \times 10^{-1}$
- $12.034 = 0.12034 \times 10^{2}$
- $0.000101 = 0.101 \times 10^{-3}$

Source of Error

Truncation Error

$\Downarrow$

Error results from estimating a formula by a formula

$\Downarrow$

TE is the difference between a truncated value $\tilde{p}$ and the actual value p arises from executing a finite number of steps to approximate an infinite process.

Exp $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^x \simeq 1 + x + \frac{x^2}{2!}$

TE = Error

$= \left| e^x - \left(1 + x + \frac{x^2}{2!}\right) \right|$

Round-off Errors

$\Downarrow$

Error results from estimating a number by a number

$\Downarrow$

Round-off Errors

Two Types

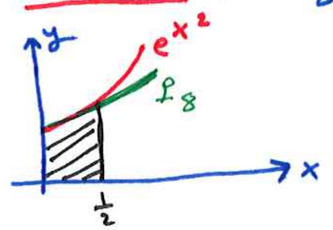
- Rounding $\Downarrow$ $fl(p)_{\text{round}}$ $\Downarrow$ rounded floating point representation
- Chopping $\Downarrow$ $fl(p)_{\text{chop}}$ $\Downarrow$ chopped floating point representation

Exp Assume the truncated Taylor series $P_8(x)$ is used to approximate $p = \int_0^{\frac{1}{2}} e^{x^2}\,dx = 0.544\,987\,104\,184$. Determine the accuracy. $\boxed{7.3}$

- $\tilde{p} = \int_0^{\frac{1}{2}} \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}\right) dx = \left(x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \frac{x^9}{216}\right)\Big|_0^{\frac{1}{2}}$

$$= 0.\underline{544\,986}\,720\,817$$

- $Z_p = \dfrac{|p-\tilde{p}|}{|p|} = \dfrac{0.000\,000\,383\,367}{0.544\,987\,104\,184}$



$$= 7.03442 \times 10^{-7}$$

$$= 0.000\,000\,703\,442 < 0.000\,005 = 5 \times 10^{-6}$$

$d = 6$

Exp ① $p = \frac{22}{7} = 3.14\,285\,714\,285\,714\,285\,7\ldots$ computer works with finite digits 16

Find the 6^{th} digits representation of p in chopping and rounding.

$fl_{chop}(p) = 3.14285 = \underline{0.314285 \times 10}$ normalized

$fl_{round}(p) = 3.14286 = \underline{0.314286 \times 10}$ normalized

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Exp Find the 4^{th} digits chopping and rounding of

② $p = 0.123444445$

$fl_{chop}(p) = 0.1234 = 0.1234 \times 10^0$

$fl_{round}(p) = 0.1235 = 0.1235 \times 10^0$

③ $y = 2.00475$

$fl_{chop}(y) = 2.004$

$fl_{round}(y) = 2.005$

④ $x = 0.000\,18279$

$fl_{chop}(x) = 0.0001827$

$fl_{round}(x) = 0.000\,1828$

Q. How does computer approximate operations?

A. Apriority to
1. Brackets
2. Powers
3. $\times$, $\div$ from left to right
4. $+$, $-$ from left to right

Exp Use 4-digits rounding to find $f(0.3456)$ if

$$f(x) = \frac{x - \sin\sqrt{x}}{2x^2 + x\cos x}$$

$$f(0.3456) = \frac{0.3456 - \sin\left(\sqrt{0.3456}\right)}{2(0.3456)^2 + (0.3456)\cos(0.3456)}$$

$$= \frac{0.3456 - \sin(0.5879)}{2(0.1194) + (0.3456)(0.9409)}$$

$$= \frac{0.3456 - 0.5546}{0.2388 + 0.3252}$$

$$= \frac{-0.2090}{0.5640}$$

$$= -0.3706$$

Exp Determine the proper answer of $\dfrac{\frac{3}{7}+\frac{5}{8}+\frac{11}{5}}{21}$ using four significant digits of accuracy.

$\frac{3}{7} = 0.428571\ldots \approx 0.4286$

$\frac{5}{8} = 0.625 = 0.6250$

$\frac{11}{5} = 2.2 = 2.200$

أولاً

$$\frac{\left(\frac{3}{7}+\frac{5}{8}\right)+\frac{11}{5}}{21} = \frac{\overbrace{0.4286+0.6250}^{1.0536}+2.200}{21}$$

$$= \frac{1.054+2.200}{21}$$

$$= \frac{3.254}{21} = 0.1550$$

0.15495

loss of Significance

- Let $p = \underline{3.14159}26536$ and
  $q = \underline{3.14159}57341$ with 11 decimal digits
- Note that $p-q = -0.0000030805$ has 5 decimal digits
- We have loss of 6 digits (which are the first 6 digits in p and q)
- This is called loss of significance or $\underbrace{\text{subtractive}}_{p-q}$ cancellation.

Exp Let $f(x) = x\left(\sqrt{x+1}-\sqrt{x}\right)$ and $g(x) = \dfrac{x}{\sqrt{x+1}+\sqrt{x}}$

Use 6 digits and rounding to compare $f(500)$ with $g(500)$.

- $f(500) = 500\left(\sqrt{501}-\sqrt{500}\right) = 500\,(22.3830 - 22.3607)$

  $= 500\,(0.0223)$ loss of 3 digits

  $= 11.1500$

- $g(500) = \dfrac{500}{\sqrt{501}+\sqrt{500}} = \dfrac{500}{22.3830+22.3607} = \dfrac{500}{44.7437} = 11.1748$

- True value is $11.1747553\cdots$ $E_f = 0.0247$ and $E_g = 0$
- Note that $g(500)$ involves less error and becomes true value to the 6 digits. so g is a better approximation than f

although $f(x) = x\left(\sqrt{x+1}-\sqrt{x}\right)\dfrac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \dfrac{x\,(x+1-x)}{\sqrt{x+1}+\sqrt{x}} = \dfrac{x}{\sqrt{x+1}+\sqrt{x}}$

$= g(x)$

هذا بسبب عملية الطرح

we solve this problem by finding $g(x)$

How to solve this function to avoid loss of significants: $\boxed{9.1}$

$\boxed{1}$ $f(x) = \ln x - \ln(x+1)$

$$f(x) = \ln\left(\frac{x}{x+1}\right)$$

$\boxed{2}$ $f(x) = \dfrac{x - \sin x}{\ln(x+2)}$. Find $f(\frac{7}{12})$ using 6 digits rounding.

- $$f(\tfrac{7}{12}) = f(0.583333) = \frac{0.583333 - \sin(0.583333)}{\ln(0.583333 + 2)}$$

$$= \frac{0.583333 - 0.550809}{0.949080}$$

$$= \frac{0.0325240}{0.949080} = 0.0342690$$

- $$f(x) = \frac{x - \sin x}{\ln(x+2)} \cdot \frac{x + \sin x}{x + \sin x} = \frac{x^2 - \sin^2 x}{[\ln(x+2)][x + \sin x]}$$

$$f(0.583333) = \frac{(0.583333)^2 - \sin^2(0.583333)}{[\ln(0.583333+2)][0.583333 + \sin(0.583333)]}$$

$$= \frac{0.340277 - 0.303391}{(0.949080)(1.13414)}$$

$$= \frac{0.0368860}{1.07639}$$

$$= 0.0342682$$

we compare with the true value

Exp Compare the results of calculating $f(0.01)$ and $P(0.01)$ using 6 digits rounding arithmetic for $\boxed{10}$

$$f(x) = \frac{e^x - 1 - x}{x^2} \quad \text{and} \quad P(x) = \frac{1}{2} + \frac{x}{6} + \frac{x^2}{24}$$

loss 1 digit — loss 2 digits

- $$f(0.01) = \frac{e^{0.01} - 1 - 0.01}{(0.01)^2} = \frac{1.01005 - 1 - 0.01}{0.0001} = \frac{0.01005 - 0.01}{0.0001}$$

$$= \frac{0.00005}{0.0001} = 0.5 \quad \Rightarrow E_f = 0.001671$$

- $$P(0.01) = \frac{1}{2} + \frac{0.01}{6} + \frac{(0.01)^2}{24}$$

P solves the problem and it is easy to find it. $E_P = 0$

$$= 0.5 + 0.00166667 + 0.00000416670 = 0.501671$$

- Note that $P(x)$ is Taylor polynomial of degree 2 for $f(x)$ at $x=0$.

  That is, $f(x) = P_2(x) + R_2(x)$.

- Now $P(0.01)$ contains less error and becomes same as true answer 0.50167084168057542... when rounding

Order of Approximation $O(h^n)$

Def • Assume $f(h)$ is approximated by the function $p(h)$.

- Assume $\exists$ a real constant $M>0$ and $\exists$ a positive integer n so that

$$* \quad \frac{|f(h) - p(h)|}{|h^n|} \le M \quad \text{for sufficiently small } h.$$

- In this case, we say $p(h)$ approximates $f(h)$ with order of approximation $O(h^n)$ and we write this as

$$f(h) = p(h) + O(h^n)$$

Note: if we write $*$ as $|f(h) - p(h)| \le M\,|h^n|$, then we see that $O(h^n)$ stands in place of the error bound $M\,|h^n|$.

Th. Assume $f(h) = p(h) + O(h^n)$ and

$g(h) = q(h) + O(h^m)$ where n, m are positive integers.

- Then $f(h) + g(h) = p(h) + q(h) + O(h^r)$

and $f(h)\,g(h) = p(h)\,q(h) + O(h^r)$

and $\dfrac{f(h)}{g(h)} = \dfrac{p(h)}{q(h)} + O(h^r)$, $g(h) \neq 0$ and $q(h) \neq 0$

where $r = \min\{n, m\}$

Exp If $f(h) = p(h) + O(h^5)$ and

$g(h) = q(h) + O(h^3)$, then $f(h)g(h) = p(h)q(h) + O(h^3)$

Remark. If $p(x)$ is the n^{th} Taylor polynomial approximation of $f(x)$, then by Taylor formula

$$f(x) = p(x) + R(x)$$

Truncation Error Term

the $\boxed{\text{remainder } R(x)}$ is simply $O(h^{n+1})$. That is

$$E = O(h^{n+1}) \approx M\,h^{n+1} \approx \frac{f^{(n+1)}(c)}{(n+1)!}\,h^{n+1}$$, h small. $c \in (x_0, x)$

Th (Taylor's Th) Assume $f \in C^{n+1}[a,b]$. Then for $x_0, x \in [a,b] \Rightarrow$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + O(h^{n+1})$$, $h = x - x_0$

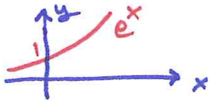
Remark ① If $k \geq n$ then $h^k + O(h^n) = O(h^n)$

Exp $h^3 + O(h^3) = O(h^3)$ since $h^3 + O(h^3) = h^3 + ch^3 = (1+c)h^3 = ch^3 = O(h^3)$

Exp $h^4 + O(h^3) = O(h^3)$

② If $f(h) = p(h) + O(h^n) = q(h) + O(h^m)$ with $n > m$, then $p(h)$ is a better approximation for $f(h)$.

Exp $e^{h} = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots$



- $e^{h} \approx 1 + h + O(h^2)$ where $E = O(h^2) \approx \frac{h^2}{2!} = \text{Error}$

order of approximation

$\boxed{e^{0.1} = 1.105170918 \quad \text{true value}}$

$e^{0.1} \approx 1 + 0.1 = 1.1$ with error $= M h^2 = M(0.1)^2 = M(0.01) < 10^{-2}$

where $M = \frac{f^{(n+1)}(c)}{(n+1)!} = \frac{e^{c}}{2!}$, $c \in (0, 0.1)$

$\Rightarrow 0 < c < 0.1 \Rightarrow 1 < e^{c} < e^{0.1} < 2 \Rightarrow M < 1$

- $e^{h} = 1 + h + \frac{h^2}{2} + O(h^3) \Rightarrow \text{Error} = O(h^3) = M h^3$

$e^{0.1} = 1 + 0.1 + \frac{0.01}{2} = 1.105 \Rightarrow \text{Error} \approx M(0.1)^3 = M(0.001) < 10^{-3}$

since $M < 1$

Exp $\sinh = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \cdots$

$\sinh \approx h$ with error $= O(h^3)$ $\Leftrightarrow \sin(0.1) \approx 0.1$

$\sinh \approx h - \frac{h^3}{3!}$ with error $= O(h^5)$ $\Leftrightarrow \sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!}$

≈ 0.0998

Exp Suppose $e^{h} = 1 + h$ $\left(\text{Error} = O(h^2)\right)$

and $\sinh = h - \frac{h^3}{3!}$ $\left(\text{Error} = O(h^5)\right)$

Then $e^{h} + \sinh = 1 + 2h - \cancel{\frac{h^3}{3!}} + O(h^2) + O(h^5)$ الأصغر هي التي تؤثر

$\approx 1 + 2h + O(h^2)$

Exp $\cosh = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \frac{h^6}{6!} + \frac{h^8}{8!} - \cdots$

$\cosh \approx 1 - \frac{h^2}{2!} + \frac{h^4}{4!}$ with $E = O(h^6) = \text{constant} \cdot h^6$

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Exp Consider the Taylor Polynomial expansions

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + o(h^4) \quad \text{and}$$

$$\cosh = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + o(h^6).$$

Determine the order of approximation for their sum and product.

- $$e^h + \cosh = 1 + h + \cancel{\frac{h^2}{2!}} + \frac{h^3}{3!} + o(h^4) + 1 - \cancel{\frac{h^2}{2!}} + \frac{h^4}{4!} + o(h^6)$$

$$= 2 + h + \frac{h^3}{3!} + o(h^4) + \frac{h^4}{4!} + o(h^6)$$

But $o(h^4) + \frac{h^4}{4!} = o(h^4)$ and

$o(h^4) + o(h^6) = o(h^4)$

$$= 2 + h + \frac{h^3}{3!} + o(h^4)$$ with order of approximation $o(h^4)$.

- $$e^h \cosh = \left(\underbrace{1 + h + \frac{h^2}{2!} + \frac{h^3}{3!}} + o(h^4)\right)\left(\underbrace{1 - \frac{h^2}{2!} + \frac{h^4}{4!}} + o(h^6)\right)$$

or

$$1 - \cancel{\frac{h^2}{2!}} + h - \frac{h^3}{2!} + \cancel{\frac{h^2}{2!}} + \frac{h^3}{3!} + o(h^4)$$

one step

$$= \left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!}\right)\left(1 - \frac{h^2}{2!} + \frac{h^4}{4!}\right) + \left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!}\right) o(h^6)$$

$$+ o(h^4)\, o(h^6) + \left(1 - \frac{h^2}{2!} + \frac{h^4}{4!}\right) o(h^4)$$

error يوجد

$$= 1 + h - \frac{h^3}{3} - \frac{5h^4}{24} - \frac{h^5}{24} + \frac{h^6}{48} + \frac{h^7}{144} + o(h^6)$$

$$+ o(h^4)\, o(h^6) + o(h^4)$$

But $o(h^4)\, o(h^6) = o(h^{10})$ and so

$$-\frac{5}{24}h^4 - \frac{h^5}{24} + \frac{h^6}{48} + \frac{h^7}{144} + o(h^6) + o(h^{10}) + o(h^4) = o(h^4)$$

Hence, $e^h \cosh = 1 + h - \frac{h^3}{3} + o(h^4)$ and the order of approximation is $o(h^4)$.

Def (order of Convergence of a sequence)

- Suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} r_n = 0$
- We say x_n converges to x with oder of convergence $O(r_n)$ if $\exists$ a constant $K>0$ s.t

$$\frac{|x_n - x|}{|r_n|} \le K \quad \text{for } n \text{ sufficiently large}$$

and we write $x_n = x + O(r_n)$

Exp show that $\boxed{1}$ $x_n = \frac{\cos n}{n^2}$ converges to 0 with rate of convergence $O(\frac{1}{n^2})$.

$$\frac{|x_n - x|}{|r_n|} = \frac{\left|\frac{\cos n}{n^2}\right|}{\left|\frac{1}{n^2}\right|} = |\cos n| \le 1 \quad \text{for all } n$$

$\boxed{2}$ $p(h) = 1 + h$ estimate $f(h) = e^h$ with order $O(h^2)$

$$\frac{|f(h) - p(h)|}{|r_n|} = \frac{|e^h - (1+h)|}{h^2} = \frac{\cancel{1} + \cancel{h} + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots - (\cancel{1+h})}{h^2}$$

$$= \frac{1}{2!} + \frac{h}{3!} + \frac{h^2}{4!} + \frac{h^3}{5!} + \cdots = \sum_{n=2}^{\infty} \frac{h^{n-2}}{n!}$$

Apply Ratio Test to see $\sum_{n=2}^{\infty} \frac{h^{n-2}}{n!}$ converges to some K

$$\lim_{n\to\infty} \frac{h^{n-1}}{(n+1)!} \frac{n!}{h^{n-2}} = \lim_{n\to\infty} \frac{h}{n+1} = 0 < 1 \quad \text{for all } h$$

$\boxed{3}$ $\sinh = h - \frac{h^3}{3!} + O(h^5)$ Exercise