

Solving non linear equations $f(x) = 0$

* In this chapter we will learn methods used to solve non linear equations numerically.

Exp ① solve $\sqrt{x} e^{\cos x} = 5 \Rightarrow \underbrace{\sqrt{x} e^{\cos x} - 5}_{f(x)} = 0$

② solve $e^x = x \Rightarrow \underbrace{e^x - x}_{f(x)} = 0$

* Numerical Methods to solve $f(x) = 0$:

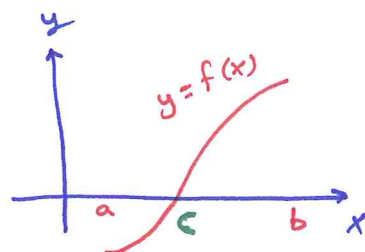
- ① Fixed Point Iteration (FPI)
- ② Bisection Method
- ③ False Position Method (or Regula Falsi Method)
- ④ Newton - Raphson Method
- ⑤ Secant Method

② and ③ are also called Bracketing Methods for locating a roots

ch2: Solution of Nonlinear Equations $f(x)=0$

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- Let $f(x) = 0$
- Assume $\exists c \in (a, b)$ s.t $f(c) = 0$
- How to estimate c ?



2.1 Iteration for Solving $x=g(x)$

Def Iteration is a repeated process until an answer is achieved.

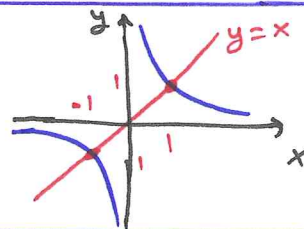
- Iteration is used to find roots of equations, solution of linear and nonlinear systems of equations, solutions of differential equations, ---
- To build an iteration, we need a rule or function $g(x)$ and a starting point P_0 .

Def (Fixed Point)

- A fixed point of a function $g(x)$ is a real number P s.t.
$$P = g(P)$$
- That is, the fixed points of $y=g(x)$ are the points of intersection of $y=g(x)$ and $y=x$

Exp ① $g(x) = \frac{1}{x} \Rightarrow$ fixed points are $1, -1$

since $P = g(P) \Leftrightarrow P = \frac{1}{P} \Leftrightarrow P^2 = 1 \Leftrightarrow P = \pm 1$



② $g(x) = x \Rightarrow$ all points are fixed points

③ $g(x) = x + 1 \Rightarrow$ No fixed points

④ $g(x) = x^3 \Leftrightarrow x = x^3 \Leftrightarrow x(x^2 - 1) = 0 \Leftrightarrow x = 0, 1, -1$

⑤ $g(x) = \cos x \Leftrightarrow x = \cos x$ "harder"

Def (Fixed Point Iteration)

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• The iteration $P_{n+1} = g(P_n)$, $n = 0, 1, 2, \dots$ is called fixed point iteration.

• That is,

P_0 starting value

$$P_1 = g(P_0)$$

$$P_2 = g(P_1)$$

\vdots

$$P_{k+1} = g(P_k)$$

\vdots

* Which types of functions $g(x)$ that produce convergent sequence $\{P_k\}$ is our main aim to study in this section.

Th* Assume g is continuous function.

• Let $\{P_n\}_{n=0}^{\infty}$ be a fixed point iteration.

• If $\{P_n\}_{n=0}^{\infty}$ converges to L " $\lim_{n \rightarrow \infty} P_n = L$ ", then L is fixed point of $g(x)$

Proof • Since $\{P_n\}_{n=0}^{\infty}$ is fixed point iteration $\Rightarrow P_{n+1} = g(P_n)$

• If $\lim_{n \rightarrow \infty} P_n = L$, then $\lim_{n \rightarrow \infty} P_{n+1} = L$.

• Hence,
$$g(L) = g\left(\lim_{n \rightarrow \infty} P_n\right) = \lim_{n \rightarrow \infty} g(P_n) = \lim_{n \rightarrow \infty} P_{n+1} = L$$

(g is cont.)

Remark • Let $f(x) = x - g(x)$

• To solve $f(x) = 0$, we solve $x = g(x)$ for fixed points.

• That is, to find the roots of $f \Rightarrow$
we find the fixed points of $g(x)$.

Exp $x^2 + 3x - 4 = 0$

17.1

- The fixed points are $(x-1)(x+4) = 0 \Leftrightarrow x = 1, -4$
- $g(x)$ can have one of the following forms:

① $g_1(x) = \frac{4-x^2}{3}$ • \Rightarrow if $P_0 = 3$ then

3 digits

$P_1 = -1.67 \Rightarrow P_2 = 0.403 \Rightarrow P_3 = 1.28 \dots \Rightarrow P_{13} = 1.005$

which converges to the fixed point 1

• \Rightarrow if $P_0 = -6$ then

$P_1 = -10.7 \Rightarrow P_2 = -36.7 \Rightarrow P_3 = -445 \dots$ diverges

• Hence, g_1 can find only the fixed point 1

② $g_2(x) = -\sqrt{4-3x}$ • \Rightarrow if $P_0 = -6$ then

$P_1 = -4.69 \Rightarrow P_2 = -4.25 \Rightarrow P_3 = -4.10 \Rightarrow \dots \Rightarrow P_{10} = -4.0000964$

which converges to the fixed point -4

③ $g_3(x) = \frac{4}{x+3}$ • \Rightarrow if $P_0 = 3$ then

$x(x+3) = 4$
 $x = \frac{4}{x+3} = g(x)$

$P_1 = 0.667 \Rightarrow P_2 = 1.09 \Rightarrow P_3 = 0.978 \dots \Rightarrow P_8 = 1.00002$

which converges to the fixed point 1

• \Rightarrow if $P_0 = -6$ then

$P_1 = -1.33 \Rightarrow P_2 = 2.40 \dots 1.004$ which converges to 1

④ $g_4(x) = x^2 + 4x - 4$

$P_3 = 0.741$
 $P_4 = 1.07 \dots$

$x^2 + 3x - 4 + x = x$
 $x^2 + 4x - 4 = x = g(x)$

it will diverge for both cases

Exp. Consider $f(x) = x^2 - 2x - 3$

• clearly the roots of $f(x) = 0 \Leftrightarrow (x-3)(x+1) = 0$
are $x = 3$ and $x = -1$

• Note that we can estimate the roots as follows:

① $x^2 - 2x - 3 = 0 \Leftrightarrow x^2 = 2x - 3 \Leftrightarrow x = \sqrt{2x + 3} = g(x)$

if $P_0 = 4$ then (If $P_0 = -\frac{3}{2}$ then $P_n \rightarrow 3$ and we could not find -)

$P_1 = g(P_0) = g(4) = \sqrt{11} \approx 3.31662$

$P_2 = g(P_1) = g(3.31662) = \sqrt{9.63324} \approx 3.10375$

$P_3 = g(P_2) = g(3.10375) = \sqrt{9.2075} \approx 3.03439$

$P_4 = g(P_3) = g(3.03439) = \sqrt{9.06878} \approx 3.01144$

\vdots
 $P_n \rightarrow 3$ (Note that 3 is fixed point of $g(x) = \sqrt{2x+3}$)

② $x(x-2) = 3 \Rightarrow x = \frac{3}{x-2} = g(x)$

If $P_0 = 4$ then

$P_1 = g(P_0) = g(4) = \frac{3}{2} = 1.5$

$P_2 = g(P_1) = g(1.5) = \frac{3}{-0.5} = -6$

$P_3 = g(P_2) = g(-6) = \frac{3}{-8} = -0.375$

$P_4 = g(P_3) = g(-0.375) = \frac{-3}{2.375} = -1.26316$

$P_5 = g(P_4) = g(-1.26316) = \frac{3}{-3.26316} = -0.91935$

$P_6 = g(P_5) = g(-0.91935) = -1.02763$

$P_7 = g(P_6) = g(-1.02763) = -0.99087$

$\vdots P_n \rightarrow -1$ (Note that -1 is fixed point of $g(x) = \frac{3}{x-2}$)

If $P_0 = -2$ then
 $\nearrow P_1 = \frac{3}{-4} = -0.75$
 $P_2 = \frac{3}{-2.75} = -1.091$
 $P_3 = \frac{3}{-3.091} = -0.971$
 $P_4 = -1.01$
 $P_5 = -0.997$
 $P_6 = -1.001$
 $P_7 = -1$
 the fixed point
 میں ممکنہ طور پر 3

③ Note that if we choose

$2x = x^2 - 3 \Rightarrow x = \frac{x^2 - 3}{2} = g(x)$ then we get a divergence sequence and so it will not work.

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$$P_0 = 4$$

$$P_1 = g(P_0) = g(4) = 6.5$$

$$P_2 = g(P_1) = g(6.5) = 19.625$$

$$P_3 = g(P_2) = g(19.625) = 191.0703$$

⋮

This is something related to the slope of the function

so it depends on how to write $g(x)$

Th (Fixed Point Theorem I) - FPTI

Assume $g \in C[a, b]$.

- If $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.
- Furthermore, if $|g'(x)| \leq K < 1$ for all $x \in (a, b)$, then g has a unique fixed point in $[a, b]$.

Proof • If $g(a) = a$ or $g(b) = b$, then we are done.

Other wise, $g(a) \in (a, b]$ and $g(b) \in [a, b)$.

Now let $f(x) = x - g(x) \Rightarrow f$ is continuous and

$$f(a) = a - g(a) < 0 \text{ and}$$

$$f(b) = b - g(b) > 0$$

Hence, by Bolzano Theorem $\exists P \in (a, b)$ s.t

$$f(P) = 0$$

$$P - g(P) = 0$$

$$P = g(P) \Rightarrow P \text{ is a fixed point.}$$

• (Uniqueness)

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suppose p_1 and p_2 are two fixed points of g

$$\Rightarrow g(p_1) = p_1 \quad \text{and} \quad g(p_2) = p_2$$

Apply Mean Value Theorem on $(p_1, p_2) \Rightarrow$

$$\exists c \in (p_1, p_2) \text{ s.t.}$$

$$g'(c) = \frac{g(p_2) - g(p_1)}{p_2 - p_1} = \frac{p_2 - p_1}{p_2 - p_1} = 1 \quad \times$$

since $|g'(x)| \leq k < 1$. Hence, $p_1 = p_2$

Exp show that $g(x) = \cos x$ has a unique fixed point in $[0, 1]$

• g is continuous on $[0, 1] \Rightarrow g \in C[0, 1]$

g is decreasing on $[0, 1]$ with

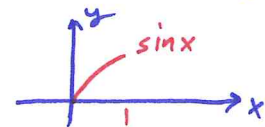
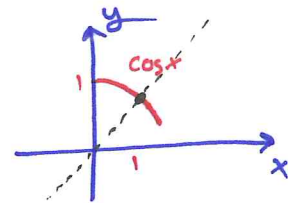
$$\cos 1 \leq \cos x \leq \cos 0$$

Hence, $g(x) \in [\cos 1, 1] \subseteq [0, 1]$ for all $x \in [0, 1]$

Thus, g has a fixed point in $[0, 1]$

• for all $x \in (0, 1) \Rightarrow |g'(x)| = |-\sin x| = \sin x \leq \sin 1 < 1$

Thus, $k = \sin 1 = \underline{0.8415} < 1 \Rightarrow g$ has a unique fixed point in $[0, 1]$.



Question: When the fixed point Iteration (FPI)

$$p_{k+1} = g(p_k) \quad , \quad p_0 \quad , \quad k = 0, 1, 2, \dots$$

produce a convergence or divergence sequence?

Note FPIT II page 21 does not apply when $g'(p) = 1$.

see Exp* page 24

Th (Fixed Point Iteration Theorem II) - FPIT II

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Assume the following :

$g, g' \in C[a, b]$, K is positive constant, $p_0 \in (a, b)$ is the starting point, $g(x) \in [a, b]$ for all $x \in [a, b]$

*• If $|g'(x)| \leq K < 1$ for all $x \in [a, b]$, then the **FPIT** converges to the unique fixed point $p \in [a, b]$.

(In this case p is called attractive fixed point)

• If $|g'(x)| > 1$ for all $x \in [a, b]$, then the **FPIT** diverges.

(In this case p is called repelling fixed point)

Remark* If p is given, then we can replace the above two conditions by :

• If $|g'(p)| < 1$, then **FPIT** converges and p is attractor

• If $|g'(p)| > 1$, then **FPIT** diverges and p is repeller

Proof We will prove *• First we prove that the points $\{p_n\}_{n=0}^{\infty} \in (a, b)$.

• since $p_0 \in (a, b) \Rightarrow$ Apply MVT on (p_0, p)

$$\textcircled{1} \dots |p - p_1| = |g(p) - g(p_0)| = |g'(c_0)(p - p_0)| = |g'(c_0)| |p - p_0| \leq K |p - p_0| < |p - p_0|$$

• Note that the assumptions above implies that $\exists_p p \in [a, b]$ by **FPIT** page 19
 $\hookrightarrow K < 1$
 $\hookrightarrow c_0 \in (a, b)$

• We see from $\textcircled{1}$ that p_1 is closer to p than $p_0 \Rightarrow p_1 \in (a, b)$

• Similar to $\textcircled{1}$ we see that $|p - p_2| < |p - p_1|$ for some $c_1 \in (a, b)$ and so $p_2 \in (a, b)$

• In general, suppose that $p_{n-1} \in (a, b) \Rightarrow$

$$\textcircled{2} \dots |p - p_n| = |g(p) - g(p_{n-1})| = |g'(c_{n-1})| |p - p_{n-1}| \leq K |p - p_{n-1}| < |p - p_{n-1}|$$

Therefore, $p_n \in (a, b)$ and hence, by induction all the points $\{p_n\}_{n=0}^{\infty} \in (a, b)$.

• Now we need to prove $\lim_{n \rightarrow \infty} |L - p_n| = 0$

Claim $|L - p_n| \leq K^n |L - p_0|$

Proof by induction when $n=1 \Rightarrow |L - p_1| \leq K |L - p_0|$ ✓ ^{See ①}

• Assume the claim holds for $n-1 \Rightarrow |L - p_{n-1}| \leq K^{n-1} |L - p_0|$

• From ② we have

$$|L - p_n| \leq K |L - p_{n-1}| \leq K K^{n-1} |L - p_0| = K^n |L - p_0|$$

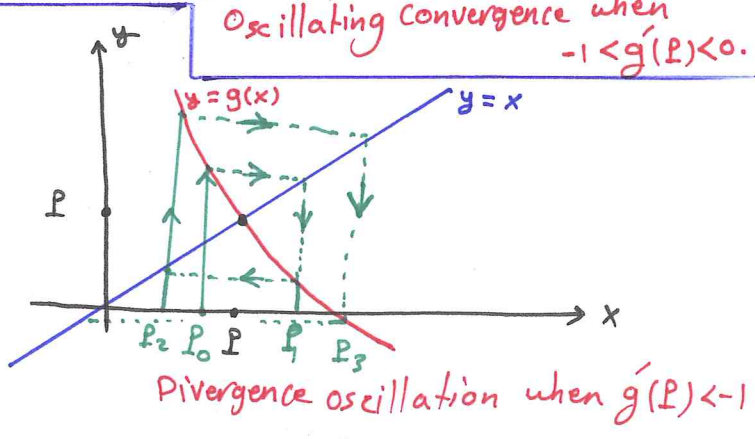
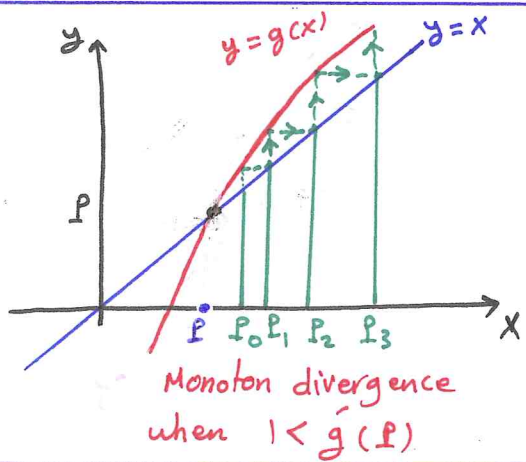
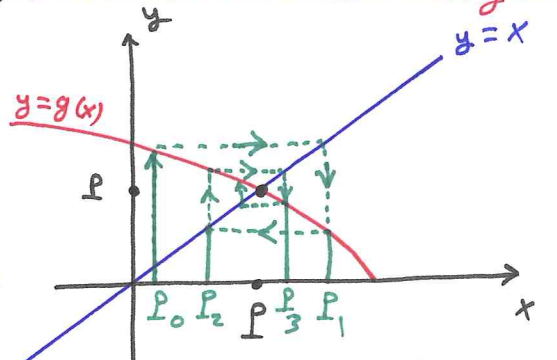
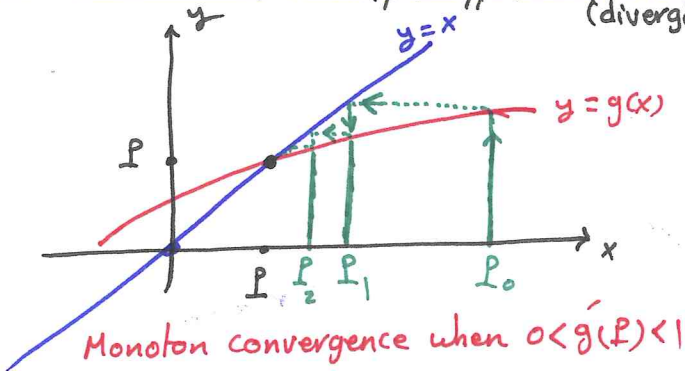
• Since $0 < K < 1 \Rightarrow \lim_{n \rightarrow \infty} K^n = 0$ • Hence,

$$0 \leq \lim_{n \rightarrow \infty} |L - p_n| \leq \lim_{n \rightarrow \infty} K^n |L - p_0| = 0$$

• Thus, by Sandwich Theorem $\lim_{n \rightarrow \infty} |L - p_n| = 0$ • So $\lim_{n \rightarrow \infty} p_n = L$ • By Th* page 17

the iteration $p_n = g(p_{n-1})$ converges to the fixed point L (since it is unique).

Two simple types of (convergent) iteration: Monoton and oscillating.



Exp Given $g(x) = \sqrt{3x-2}$.

22.1

Find the fixed points of $g(x)$ and determine the nature of these fixed points.

$$\begin{aligned}x = g(x) &\Leftrightarrow x = \sqrt{3x-2} && \Leftrightarrow x^2 = 3x-2 \\&\Leftrightarrow x^2 - 3x + 2 = 0 && \Leftrightarrow (x-1)(x-2) = 0 \\&\Leftrightarrow x=1, x=2 \text{ are fixed points}\end{aligned}$$

$$\begin{aligned}g'(x) = \frac{3}{2\sqrt{3x-2}} &\Rightarrow |g'(1)| = \frac{3}{2} > 1 \Rightarrow x=1 \text{ is repeller} \\&\Rightarrow \text{FPI diverges}\end{aligned}$$

$$\begin{aligned}\Rightarrow |g'(2)| = \frac{3}{4} < 1 &\Rightarrow x=2 \text{ is attractor} \\&\Rightarrow \text{FPI converges to 2}\end{aligned}$$

Exp $x^2 + 3x - 4 = 0$ has fixed points $x=1, -4$ 22.2

$$\textcircled{1} \quad g_1(x) = \frac{4-x^2}{3}$$

$$g_1'(x) = -\frac{2}{3}x \Rightarrow$$

• $|g_1'(1)| = |-\frac{2}{3}| < 1 \Rightarrow x=1$ is attractor

\Rightarrow FPI converges to 1. That is we can use the FPI to find the solution of $x=g(x)$. (see page 17.1)

• $|g_1'(-4)| = |\frac{8}{3}| > 1 \Rightarrow x=-4$ is repeller

\Rightarrow FPI diverges (see page 17.1)

$$\textcircled{2} \quad g_2(x) = -\sqrt{4-3x} \Rightarrow g_2'(x) = \frac{3}{2\sqrt{4-3x}}$$

• $g_2'(1) = |\frac{3}{2}| > 1 \Rightarrow x=1$ is repeller ($P_0 \neq 2$)
 \Rightarrow FPI does not work (see page 17.1)

• $g_2'(-4) = |\frac{3}{8}| < 1 \Rightarrow x=-4$ is attractor
 \Rightarrow FPI converges to -4 (see page 17.1)

see also
Exp page
17.1

Exp Test the convergence of FPI for $\textcircled{1} \quad g(x) = -4 + 4x - \frac{x^2}{2}$

• $g(x) = x \Leftrightarrow -4 + 3x - \frac{x^2}{2} = 0 \Leftrightarrow x^2 - 6x + 8 = 0 \Leftrightarrow x = 2, 4$ fixed points

• $g'(x) = 4 - x \Rightarrow g'(2) = |4-2| = 2 > 1 \Rightarrow x=2$ is repeller \Rightarrow FPI div.
 $\Rightarrow g'(4) = |4-4| = 0 < 1 \Rightarrow x=4$ is attractor \Rightarrow FPI conv.

page 24 $\textcircled{2} \quad g(x) = 2\sqrt{x-1} \Leftrightarrow x = g(x) \Leftrightarrow x^2 = 4(x-1) \Leftrightarrow x^2 - 4x + 4 = 0 \Leftrightarrow x = 2$ fixed point

$g'(x) = \frac{1}{\sqrt{x-1}} \Rightarrow g'(2) = |1| = 1$ Test Fails so we build FPI to check (see Exp* page 24).

Exp Let $g(x) = 1 + x - \frac{x^2}{4}$. Can we use the **FPI** 23 to find the solution of the equation $x = g(x)$? Why?

Solution • $x = g(x) \Leftrightarrow x = 1 + x - \frac{x^2}{4} \Leftrightarrow x^2 = 4 \Leftrightarrow \boxed{x = \pm 2}$ Fixed Points

• $\boxed{x = 2} \Rightarrow g'(x) = 1 - \frac{x}{2} \Rightarrow |g'(2)| = 0 < 1$ ↖ K for x=2

Hence, by **Remark** page 21 \Rightarrow the **FPI** converges to 2

• So we can use the **FPI** to find the solution of $x = g(x)$:

$P_0 = 1.6$

$P_1 = g(P_0) = g(1.6) = 1 + 1.6 - \frac{(1.6)^2}{4} = 1.96$

$P_2 = g(P_1) = g(1.96) = 1 + 1.96 - \frac{(1.96)^2}{4} = 1.9996$

$P_3 = g(P_2) = g(1.9996) = 1 + 1.9996 - \frac{(1.9996)^2}{4} = 2$

$P_n \rightarrow 2$

↑
attractor
fixed point

$P_0 = 2.5$

$P_1 = g(P_0) = g(2.5) = 1 + 2.5 - \frac{(2.5)^2}{4} = 1.9375$

$P_2 = g(P_1) = g(1.9375) = 1.999$

$P_3 = g(P_2) = g(1.999) = 2$

$P_n \rightarrow 2$

• $\boxed{x = -2}$

$\Rightarrow |g'(-2)| = |1 - \frac{-2}{2}| = 2 > 1$. Hence, by **Remark***

FPI diverges and $P = -2$ is repeller (repulsive) fixed point.

To see that:

$P_0 = -2.05$

$P_1 = g(P_0) = g(-2.05) = 1 - 2.05 - \frac{(-2.05)^2}{4} = -2.1$

$P_2 = g(P_1) = g(-2.1) = -2.2025$

$P_3 = g(P_2) = g(-2.2025) = -2.4153$

\vdots
 P_n diverges

Exp* Consider the iteration $p_{n+1} = g(p_n)$ where $g(x) = 2\sqrt{x-1}$ 24
for $x \geq 1$. Can **FPI** be used to find the solution of $x = g(x)$?

Fixed point

$$\begin{aligned}x &= g(x) \Leftrightarrow x = 2\sqrt{x-1} \Leftrightarrow x^2 = 4(x-1) \\ \Leftrightarrow x^2 - 4x + 4 &= 0 \Leftrightarrow (x-2)(x-2) = 0 \Leftrightarrow \boxed{x=2}\end{aligned}$$

FPI

$$\hat{g}(x) = \frac{1}{\sqrt{x-1}} \Rightarrow \hat{g}(2) = 1$$

\Rightarrow **FPIITII** does not apply

There are two cases to consider:

case 1 start with $p_0 = 1.5$

$$p_1 = g(p_0) = g(1.5) = 2\sqrt{1.5-1} = 1.4142$$

$$p_2 = g(p_1) = g(1.4142) = 1.2872$$

$$p_3 = g(p_2) = g(1.2872) = 1.0718$$

$$p_4 = g(p_3) = g(1.0718) = 0.5359 \quad \text{outside the domain of } g$$

p_5 can not be computed

case 2 start with $p_0 = 2.5$

$$p_1 = g(p_0) = g(2.5) = 2\sqrt{2.5-1} = 2.4495$$

$$p_2 = g(p_1) = g(2.4495) = 2.4079$$

$$p_3 = g(p_2) = g(2.4079) = 2.3731$$

$$p_4 = g(p_3) = g(2.3731) = 2.3436$$

\vdots

$$\lim_{n \rightarrow \infty} p_n = 2 \quad \text{"slowly"} \Rightarrow p_{1000} = 2.004$$

Hence, the FPI converges in this exp for every $p_0 > 2$
and diverges for every $p_0 \leq 2$

Corollary Assume g satisfies conditions of FPIT II. 25

If p_n is used to approximate the fixed point L , then an upper bounds for the error are

$$|L - p_n| \leq K^n |L - p_0| \quad \text{for all } n \geq 1 \quad \dots \textcircled{A}$$

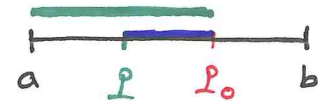
and

$$|L - p_n| \leq \frac{K^n}{1-K} |p_1 - p_0| \quad \text{for all } n \geq 1 \quad \dots \textcircled{B}$$

Proof
(HW)
see page
25.1

• \textcircled{A} was the claim in the proof of FPIT II and has been proven. (K is the upper bound of $g'(x)$)

• To prove \textcircled{B} we use \textcircled{A} as follows:

$$|L - p_n| \leq K^n |L - p_0| \leq K^n \max\{p_0 - a, b - p_0\}$$


• For $n \geq 1$ we have

$$* |p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \leq K |p_n - p_{n-1}| \leq K^n |p_1 - p_0|$$

• Therefore, for $m > n \geq 1$ we have

$$\begin{aligned} |p_m - p_n| &= |p_m - p_{m-1} + p_{m-1} + \dots + p_{n+1} - p_n| \\ &\leq |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \dots + |p_{n+1} - p_n| \\ &\leq K^{m-1} |p_1 - p_0| + K^{m-2} |p_1 - p_0| + \dots + K^n |p_1 - p_0| \quad \text{by } * \\ &= (1 + K + K^2 + \dots + K^{m-n-1}) K^n |p_1 - p_0| = K^n |p_1 - p_0| \sum_{i=0}^{m-n-1} K^i \end{aligned}$$

• Since $\lim_{m \rightarrow \infty} p_m = L$ it follows that

$$|L - p_n| = \lim_{m \rightarrow \infty} |p_m - p_n| \leq$$

$m > n \Rightarrow K^m < K^n$
since $K < 1$

$$K^n |p_1 - p_0| \sum_{i=0}^{\infty} (K)^i = \frac{K^n}{1-K} |p_1 - p_0|$$

Proof of (B) in different way :

$$\begin{aligned} \bullet \quad |P - P_0| &= |P - P_1 + P_1 - P_0| \\ &\leq |P - P_1| + |P_1 - P_0| \quad \text{by Triangle Inequality} \\ &\leq K |P - P_0| + |P_1 - P_0| \quad \text{by (A)} \end{aligned}$$

$$\bullet \quad \text{Hence, } |P - P_0| - K |P - P_0| \leq |P_1 - P_0|$$

$$(1 - K) |P - P_0| \leq |P_1 - P_0|$$

$$|P - P_0| \leq \frac{1}{1 - K} |P_1 - P_0| \quad \dots *$$

• From (A) we have

$$\begin{aligned} |P - P_n| &\leq K^n |P - P_0| \\ &\leq K^n \frac{1}{1 - K} |P_1 - P_0| \\ &= \frac{K^n}{1 - K} |P_1 - P_0| \end{aligned}$$

Remark • This Corollary provides stopping criteria to the FPI

• That is, it tells us the number of iterations n for a given upper bound of the error.

Exp Let $x^3 - x - 5 = 0$

① show that $f(x)$ has a root on $[0, 2]$

$$f(x) = x^3 - x - 5 \quad \Rightarrow \quad \left. \begin{array}{l} f(0) = -5 < 0 \\ f(2) = 1 > 0 \end{array} \right\} \Rightarrow \text{By Bolzano} \\ f \text{ cont. on } [0, 2] \quad \exists \text{ a root on } [1, 2]$$

② Consider $g(x) = \sqrt[3]{x+5}$. Use 5 decimal with $p_0 = 1.5$ to approximate the root of $f(x)$ "FP of $g(x)$ " by finding p_1, p_2, p_3

$$p_1 = g(p_0) = g(1.5000) = \sqrt[3]{6.5000} = 1.8663$$

$$p_2 = g(p_1) = g(1.8663) = \sqrt[3]{6.8663} = 1.9007$$

$$p_3 = g(p_2) = g(1.9007) = \sqrt[3]{6.9007} = 1.9038$$

FP
 $P = 1.9042$
 True
 $P = -0.9521 \pm 1.3113i$
 Complex roots

③ Find K

K is upper bound for $|g'(x)|$ on $[0, 2]$

$$|g'(x)| = \left| \frac{1}{3 \sqrt[3]{(x+5)^2}} \right| = \frac{1}{3 \sqrt[3]{(x+5)^2}} \leq \frac{1}{3} \frac{1}{\sqrt[3]{25}} \text{ since } g' \text{ decreasing on } [0, 2]$$

$$|g'(x)| \leq \frac{1}{3} \frac{1}{\sqrt[3]{25}} \leq \frac{1}{3} \frac{1}{\sqrt[3]{8}} = \frac{1}{3} \frac{1}{2} = \frac{1}{6} < 1$$

K

$K_{\max} = 0.114$

$$K_{\min} = \frac{1}{3 \sqrt[3]{49}} = 0.0911$$

④ show that $g(x)$ has a unique fixed point in $[0, 2]$

- $g(x)$ is cont. on $[0, 2]$
- $g(x)$ is increasing on $[0, 2]$ $\Rightarrow \begin{matrix} \sqrt[3]{5} = g(0) \leq g(x) \leq g(2) = \sqrt[3]{7} \leq \sqrt[3]{8} \\ 0 \leq g(x) \leq 2 \quad \forall x \in [0, 2] \end{matrix}$

Hence, by FP II g has a fixed point in $[0, 2]$

- $|g'(x)| = \frac{1}{3 \sqrt[3]{(x+5)^2}} \leq K = \frac{1}{6} < 1$ since g' is decreasing

g has a unique fixed point in $[0, 2]$

Exp Consider $g(x) = \frac{1}{x^3} + 2$ on $[2, 3]$

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① show that $g(x)$ has fixed points in $[2, 3]$

• Note that g is continuous on $[2, 3]$

• Note also g is decreasing on $[2, 3]$

$$2 < 2.037 = g(3) \leq g(x) \leq g(2) = 2.125 < 3 \quad \forall x \in [2, 3]$$

• Hence, $g(x) \in [2, 3] \quad \forall x \in [2, 3]$

• Therefore by **FPTI** page 19 g has a fixed point in $[2, 3]$

② Is it unique?

• $|g'(x)| = \left| -\frac{3}{x^4} \right| = \frac{3}{x^4}$ decreasing on $(2, 3)$

• $|g'(x)| \leq \frac{3}{(2)^4} = 0.1875 = K_{\max} < 1$ for all $x \in (2, 3)$

• Hence, g has a unique fixed point in $[2, 3]$

③ show that the FPI converges for every $P_0 \in (2, 3)$

• Note that $g, g' \in C[2, 3]$ and $g(x) \in [2, 3] \quad \forall x \in [2, 3]$

• Note also $|g'(x)| \leq 0.1875 = K < 1 \quad \forall x \in [2, 3]$

• Hence, by **FPII** page 21 the FPI converges $\forall P_0 \in (2, 3)$

④ Using 4 digits rounding, estimate the fixed point

of $g(x)$ using $P_0 = 1.5$ and with error less

than 0.001.

n	P_n	Upper bound of the error
0	1.500	—
1	2.296	0.1838 > 0.001
2	2.083	0.03445 > 0.001
3	2.111	0.006458 > 0.001
4	2.106	0.001211 > 0.001
5	2.107	0.0002270 < 0.001

$$P_{n+1} = \frac{1}{P_n^3} + 2 \quad \boxed{28}$$

$$|P - P_n| \leq \frac{K^n}{1-K} |P_1 - P_0|$$

We can use the idea of $\boxed{5}$ below here too, by finding n first see page 36

We can use $|P_n - P_{n-1}|$ as stopping criteria

$$P_1 = \frac{1}{P_0^3} + 2 = \frac{1}{(1.5)^3} + 2 = 2.296$$

$$P_2 = \frac{1}{P_1^3} + 2 = \frac{1}{(2.296)^3} + 2 = 2.083$$

⋮

$$E_1 \leq \frac{K}{1-K} |P_1 - P_0| = \frac{0.1875}{1-0.1875} |2.296 - 1.500| = 0.1838$$

$$E_2 \leq \frac{K^2}{1-K} |P_1 - P_0| = \frac{(0.1875)^2}{1-0.1875} |2.296 - 1.500| = 0.03445$$

⋮

$\boxed{5}$ Find number of iterations required to estimate the fixed point of $g(x)$ with accuracy of 10^{-5} .

$$|P - P_n| \leq \frac{K^n}{1-K} |P_1 - P_0| < 10^{-5}$$

$$\frac{(0.1875)^n}{1-0.1875} |2.296 - 1.500| < 10^{-5}$$

$$\frac{(0.1875)^n}{0.8125} (0.7960) < 10^{-5}$$

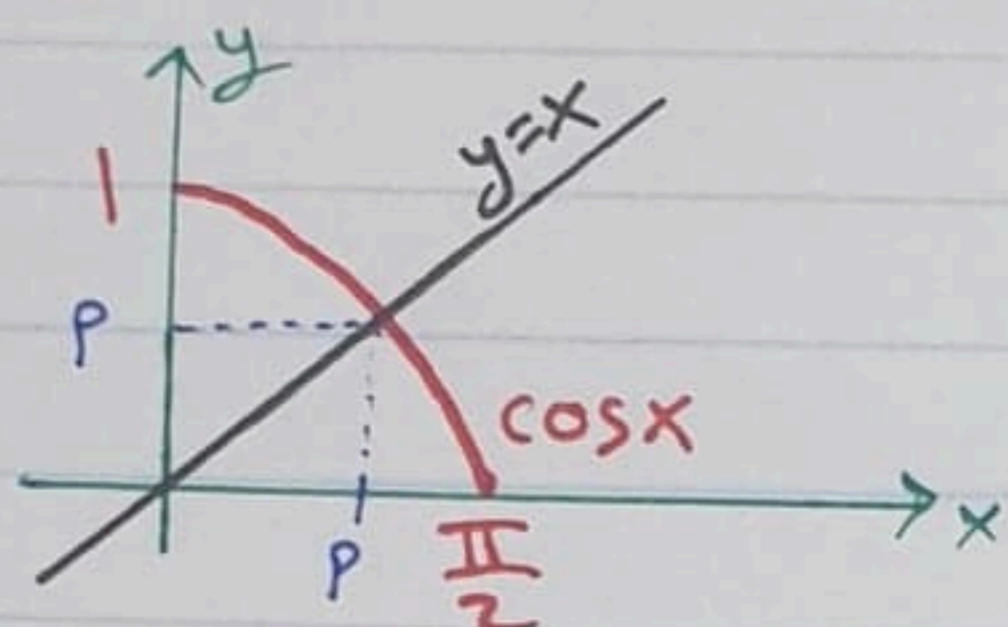
$$(0.1875)^n (0.9797) < 10^{-5}$$

$$(0.1875)^n < 1.021 \times 10^{-5} = 0.00001021 \Leftrightarrow \boxed{n=7}$$

Exp Let $g(x) = \cos x$ on $[0, 1]$. Use 4 digits to

(1) Show that g has a unique fixed point in $[0, 1]$

- g is cont. on $[0, 1]$
- g is decreasing on $[0, 1]$ so $g(1) \leq g(x) \leq g(0)$



$$0 \leq 0.5403 \leq g(x) \leq 1$$

Hence, $g(x) \in [0, 1] \forall x \in [0, 1]$
Therefore, by **FPT I** g has a fixed point in $[0, 1]$

- To prove the fixed point is **unique** \Rightarrow

$$|g'(x)| = |-\sin x| = \sin x \leq \sin 1 = 0.8415 = K < 1 \quad \forall x \in (0, 1)$$

\rightarrow since $\sin x$ is increasing on $[0, 1]$

Hence, g has a unique FP in $[0, 1]$

(2) Find the number of iterations needed to estimate this FP of g with error of magnitude less than 10^{-3} (take $p_0 = 0.5$)

$$p_1 = g(p_0) = \cos(0.5) = 0.8776 \quad \Rightarrow |p_1 - p_0| = 0.3776$$

$$1 - K = 1 - 0.8415 = 0.1585$$

$$|p - p_n| \leq \frac{K^n}{1-K} |p_1 - p_0| < 10^{-3}$$

$$\frac{(0.8415)^n}{0.1585} (0.3776) < 10^{-3}$$

$$(0.8415)^n < \frac{10^{-3}}{2.382} = 0.0004198$$

$$(0.8415)^n < 0.0004198$$

$$n \geq 46$$