Th (Accelerated Newton-Raphson Iteration)

- Assume Newton - Raphson iteration

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}, \text{ given } P_0, \ n=0,1,2,\cdots$$

produces a sequence $\{P_n\}$ that converges to the root p.

- Assume p is a multiple root (of order $M>1$). Then

[1] Done " by Th page 45 $\Rightarrow$ Newton's iteration $\{P_n\}$ converges linearly $(R=1)$ with $A = \frac{M-1}{M}$ and $\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|} = A$ "

[2] the modification of Newton's iteration

$$P_{n+1} = P_n - \frac{M\, f(P_n)}{f'(P_n)}, \text{ given } P_0, \ n=0,1,2,\cdots$$

converges quadratically $(R=2)$ to p and $A = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2}$

Exp $f(x) = x^3 - 3x + 2$. Estimate $p=1$ using accelerated newton method with $P_0 = 1.2$

- Note that $p=1$ has multiplicity $M=2$ since $f(1) = f'(1) = 0$ but $f''(1) = 6 \neq 0$
- Acceleration formular: $P_{n+1} = P_n - \frac{2f(P_n)}{f'(P_n)} = \frac{P_n^3 + 3P_n - 4}{3P_n^2 - 3}$

n	P_n	$E_n = \lvert p - P_n \rvert$	$\lvert E_{n+1} \rvert / \lvert E_n \rvert^2$
0	1.2	0.2	0.151515150
1	1.006060606	0.006060606	0.165718578 $\approx A$
2	1.000006087	0.000006087	
3	1	0	

Proof of Th (Accelerated Newton-Raphson Iteration) $\boxed{50.1}$

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We need to show if p is multiple root $(M>1)$, then the accelerated iteration

$$P_{n+1} = P_n - \frac{M f(P_n)}{f'(P_n)}, \quad \text{given } P_0, \quad n = 0,1,2,\ldots$$

converges quadratically $(R=2)$ to the root P.

- Since p is multiple $(M>1) \Rightarrow f(x) = (x-p)^M h(x)$ where $h(x)$ is cont. s.t $h(p) \neq 0$.

- Define $\boxed{g(x) = x - \frac{M f(x)}{f'(x)} = x - \frac{M(x-p)h(x)}{M h(x) + (x-p)h'(x)}}$

see page 49.3

$$g'(P) = 1 - \frac{M^2 h^2(P)}{M^2 h^2(P)}$$

see page 49.4

$$= 1 - 1$$

$$g'(P) = 0$$

- Expand g about P using Taylor expansion

$$g(x) = g(P) + g'(P)(x-P) + \frac{g''(c)}{2!}(x-P)^2, \quad c \in (x,P)$$

$$g(P_n) = P + 0\,(P_n - P) + \frac{g''(c)}{2!}(P_n - P)^2, \quad c \in (P_n, P)$$

$$P_{n+1} = P + \frac{g''(c)}{2}(P_n - P)^2$$

$P_n < c < P$

$$P_{n+1} - P = \frac{g''(c)}{2}(P_n - P)^2$$

since $P_n \to P$ as $n \to \infty$

$c \to P$ as $n \to \infty$

$$E_{n+1} = \frac{g''(c)}{2} E_n^2$$

$$\frac{E_{n+1}}{E_n^2} = \frac{g''(c)}{2}$$

$$\frac{|E_{n+1}|}{|E_n|^2} = \frac{1}{2} |g''(c)|$$

$$\lim_{n \to \infty} \frac{|E_{n+1}|}{|E_n|^2} = \frac{1}{2} \lim_{n \to \infty} |g''(c)|$$

$$A = \frac{1}{2} |g''(p)|$$

$R = 2$ since $g''(p) \neq 0$

Remark: Comparing the Exp page 47 using Newton Iteration with same Exp page 50 we see that using Accelerated Newton Iteration we need only 4 iterations to reach the exact root.

Exercises: solve questions 17, 18, 21, 23 page 86, 87, 88