Speed of Convergence for FPI

$\boxed{50.3}$

Exp show that if $g(p) = p$ and $g'(p) = g''(p) = 0$, then the fixed point iteration converges to p with R at least 3 and $A = \frac{1}{6}|g'''(p)|$

- Recall the fixed Point Iteration $p_{n+1} = g(p_n)$
- Expand g about the root p

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \frac{g'''(c)}{3!}(x-p)^3$$

$$g(p_n) = p + 0 + 0 + \frac{g'''(c)}{6}(p_n - p)^3$$

$$p_{n+1} = p + \frac{g'''(c)}{6}(p_n - p)^3$$

$$p_{n+1} - p = \frac{g'''(c)}{6}(p_n - p)^3$$

$x < c < p$

$p_n < c < p$

$c \to p$ as $n \to \infty$

since $p_n \to p$ as $n \to \infty$

$$E_{n+1} = \frac{|g'''(c)|}{6} E_n^3$$

$$\frac{|E_{n+1}|}{|E_n|^3} = \frac{1}{6}|g'''(c)|$$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^3} = \frac{1}{6}\lim_{n\to\infty}|g'''(c)| = \frac{1}{6}|g'''(p)|$$

$A = \frac{1}{6}|g'''(p)|$, R at least 3 since

if $g'''(p) \neq 0 \Rightarrow R = 3$

if $g'''(p) = 0 \Rightarrow R > 3$

Exp Let p be a fixed point of $g(x)$.
show that if $g'(p) = g''(p) = \cdots = g^{(k-1)}(p) = 0$ and
$g^{(k)}(p) \neq 0$, then the fixed point iteration of $g(x)$
will converge to p with

$$R = k \quad \text{and} \quad A = \left| \frac{g^{(k)}(p)}{k!} \right|$$

Apply Taylor Series of $g(x)$ about p $\Rightarrow$

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \cdots + \frac{g^{(k-1)}(p)}{(k-1)!}(x-p)^{k-1} + \frac{g^{(k)}(c)}{k!}(x-p)^k$$

$$g(p_n) = p + 0 + 0 + \cdots + 0 + \frac{g^{(k)}(c)}{k!}(p_n - p)^k$$

$$p_{n+1} = p + \frac{g^{(k)}(c)}{k!}(p_n - p)^k \qquad \text{since } p_{n+1} = g(p_n) \text{ is FPI}$$

$$p_{n+1} - p = \frac{g^{(k)}(c)}{k!}(p_n - p)^k$$

$$E_{n+1} = \frac{\left|g^{(k)}(c)\right|}{k!} E_n^k$$

$$\frac{E_{n+1}}{E_n^k} = \frac{1}{k!}\left|g^{(k)}(c)\right|$$

$x < c < p$

$p_n < c < p$

$c \to p$ as $n \to \infty$ since
$p_n \to p$ as $n \to \infty$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^k} = \frac{1}{k!} \lim_{n\to\infty} \left|g^{(k)}(c)\right| = \frac{1}{k!}\left|g^{(k)}(p)\right|$$

$$A = \left| \frac{g^{(k)}(p)}{k!} \right| \quad \text{with} \quad R = k$$

Remark: Newton's iteration is special case of FPI with
$g(x) = x - \frac{f(x)}{f'(x)}$. compare page 49.1 with 50.4