Speed of Convergence for Secant Method

$$P_{n+2} = P_{n+1} - \left(\frac{P_{n+1} - P_n}{f(P_{n+1}) - f(P_n)} \right) f(P_{n+1}) \qquad \text{given } P_0, P_1$$

1. if p is simple root $(M=1)$, then secant's iteration $\{P_n\}$ converges to p with

$$R = 1.618 \quad \text{and} \quad \lim_{n \to \infty} \frac{|E_{n+1}|}{|E_n|^{1.618}} = A = \left| \frac{f''(p)}{2f'(p)} \right|^{0.618}$$

2. if p is multiple root (of order $M>1$), then secant's iteration converges to p with

$$R = 1 \quad \text{and} \quad A \text{ depends on } f(x)$$

Exp Start with $P_0 = -2.6$ and $P_1 = -2.4$ and use the secant method to

1. find the root $p=-2$ of $f(x) = x^3 - 3x + 2$
2. find the order of convergence R for $p=-2$
3. find the asymptotic error constant A for $p=-2$
4. Prove part 3 numerically.

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2. Recall that $p=-2$ is simple root since $f(-2)=0$ but $f'(-2) = 9 \neq 0$. Hence, $R = 1.618$

3. $$A = \left| \frac{f''(-2)}{2f'(-2)} \right|^{0.618} = \left(\frac{2}{3}\right)^{0.618} = 0.778351205$$

n	p_n	$E_n = \lvert p - p_n \rvert$	$\lvert E_{n+1}\rvert / \lvert E_n \rvert^{1.618}$
0	-2.6	0.6	0.914152831
1	-2.4	0.4	0.469497765
2	-2.106598985	0.106598985	0.847290012
3	-2.022641412	0.022641412	0.693608922
4	-2.001511098	0.001511098	0.825841116
5	-2.000022537	0.000022537	0.727100987 $\approx A$
6	-2.000000022	0.000000022	
7	-2	0	

$$1.618 \approx \frac{1+\sqrt{5}}{2}$$

- This exp shows the convergence of the secant method at simple root $p = -2$
- Note that $E_5 = |p - p_5| = 0.000022537$

$$E_4^{1.618} = |p - p_4|^{1.618} = (0.001511098)^{1.618} = 0.000027296$$

- It is easy to check that $|E_5| \approx A \, |E_4|^{1.618}$ $\Leftarrow$

$$0.000022537 \approx (0.778351205)(0.000027296)$$

$$= 0.0000212459$$

- Speed of Convergence for Bisection Method: $R = 1$ and $A = \frac{1}{2}$

$$\frac{|E_{n+1}|}{|E_n|} \approx \frac{1}{2}$$

- Speed of Convergence for False Position Method: $R = 1$ and A depends on $f(x)$ $\Rightarrow \dfrac{|E_{n+1}|}{|E_n|} \approx A$