2.2 Bracketing Methods for Locating a Root

Def. • Assume $f(x)$ is continuous function.
• The number r is called a root of the equation $f(x)=0$ iff $f(r)=0$.
or zero

Exp $2x^2+3x-2=0$ has two roots $r_1=\frac{1}{2}$ and $r_2=-2$

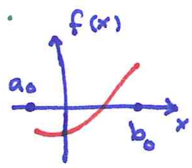
$x^2+\frac{3}{2}x-1=0 \iff (x-\frac{1}{2})(x+2)=0$

In this section we will learn two Bracketing methods for finding a zero of a continuous function $f(x)=0$:

1. Bisection Method of Bolzano
2. False Position Method

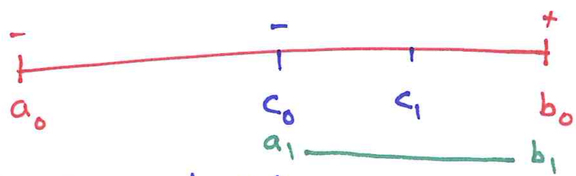
Bisection Method of Bolzano

• This method used to solve $f(x)=0$.
• This method depends on Bolzano Theorem.



• Conditions required • $f \in C[a,b]$ and
• $f(a)\,f(b)<0$

• Let $[a,b]=[a_0,b_0]$



• First Iteration $c_0=\dfrac{a_0+b_0}{2}$ is the midpoint

• Now find $f(c_0)$. We have three cases (c_1 is 2^{nd} iteration)

If $f(c_0)=0$, then c_0 is the root and we are done.

If $f(c_0)\,f(a_0)<0$, then $[a_1,b_1]=[a_0,c_0]$ and $c_1=\dfrac{a_1+b_1}{2}=\dfrac{a_0+c_0}{2}$

If $f(c_0)\,f(b_0)<0$, then $[a_1,b_1]=[c_0,b_0]$ and $c_1=\dfrac{a_1+b_1}{2}=\dfrac{c_0+b_0}{2}$

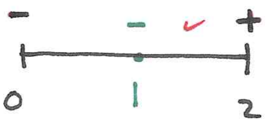
- In general, the $n+1^{th}$ iteration is $\boxed{c_n = \frac{a_n + b_n}{2}}$ in Bisection Method where $n = 0, 1, 2, \ldots$

Exp Use the Bisection Method to find the first five iterations c_0, c_1, c_2, c_3, c_4 that estimate the root of $x \sin x - 1 = 0$ on $[0, 2]$. "Use 4 digits"

- $f(x) = x \sin x - 1$ $\quad [a_0, b_0] = [0, 2]$ 1^{st} interval

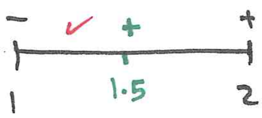
- $f(0) = -1 < 0$ and $f(2) = 0.8180 > 0$

  1^{st} iteration $c_0 = \frac{a_0 + b_0}{2} = \frac{0+2}{2} = 1$



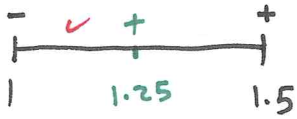
- $f(c_0) = f(1) = -0.1585 < 0 \Rightarrow [a_1, b_1] = [1, 2]$ 2^{nd} interval

  2^{nd} iteration $c_1 = \frac{a_1 + b_1}{2} = \frac{1+2}{2} = 1.5$



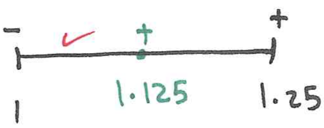
- $f(c_1) = f(1.5) = 0.4962 > 0 \Rightarrow [a_2, b_2] = [1, 1.5]$ 3^{rd} interval

  3^{rd} iteration $c_2 = \frac{a_2 + b_2}{2} = \frac{1 + 1.5}{2} = 1.25$



- $f(c_2) = f(1.25) = 0.1862 > 0 \Rightarrow [a_3, b_3] = [1, 1.25]$ 4^{th} interval

  4^{th} iteration $c_3 = \frac{a_3 + b_3}{2} = \frac{1 + 1.25}{2} = 1.125$



- $f(c_3) = f(1.125) = 0.01505 > 0 \Rightarrow [a_4, b_4] = [1, 1.125]$ 5^{th} interval

  5^{th} iteration $c_4 = \frac{a_4 + b_4}{2} = \frac{1 + 1.125}{2} = 1.063$

- $f(c_4) = f(1.063) = -0.07183$

Note $\sin x$ is evaluated in radians $\Rightarrow$

$f(2) = 2 \sin(114.6) - 1 = 2(0.9092) - 1$

$= 1.818 - 1 = 0.8180$

$\pi \rightarrow 3.14$

$? \rightarrow 2$

$? = 0.6369\pi$ $\leftarrow$ 180

$= 114.6$

Th (Bisection Theorem)

- Assume that $f \in C[a,b]$,

  $\exists$ a number $r \in [a,b]$ s.t $f(r) = 0$,

  $f(a)f(b) < 0$ and $\{c_n\}_{n=0}^{\infty}$ represents the sequence of midpoints generated by the bisection method.

- Then an upper bound of the error is

$$|r - c_n| \le \frac{b-a}{2^{n+1}} \quad \text{for } n = 0, 1, 2, \cdots$$

- Furthermore, the sequence $\{c_n\}_{n=0}^{\infty}$ converges to the zero r. That is, $\lim_{n\to\infty} c_n = r$.

Proof

- Let $[a_0, b_0] = [a, b]$

- Note that

$$b_1 - a_1 = \frac{b_0 - a_0}{2} = \frac{b-a}{2}$$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b-a}{2^2}$$

see Exp in page 31

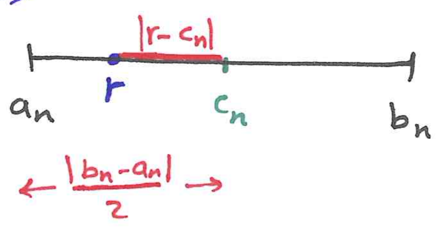
$$b_3 - a_3 = \frac{b_2 - a_2}{2} = \frac{b-a}{2^3}$$

$$\vdots$$

$$\boxed{b_n - a_n = \frac{b-a}{2^n}} \quad *$$

- Now since both the zero r and the midpoint $c_n = \frac{b_n - a_n}{2}$ lie in the interval $[a_n, b_n] \Rightarrow$

$$|r - c_n| \le \frac{b_n - a_n}{2} \quad \text{for all } n$$



- Using $*$ we obtain

$$|r - c_n| \le \frac{b-a}{2^{n+1}} \quad \text{for all } n$$

- To prove the second part, note that

$$0 \le |r - c_n| \le \frac{b-a}{2^{n+1}}$$

and $\lim\limits_{n\to\infty} \frac{b-a}{2^{n+1}} = 0$. Hence, by Sandwich Theorem

$$\lim_{n\to\infty} |r - c_n| = 0 \iff \lim_{n\to\infty} (r - c_n) = 0$$

$$\iff r = \lim_{n\to\infty} c_n$$

Remark: The Bisection Theorem provides a strategy to find the number of iteration for a given accuracy δ:

$$\frac{b-a}{2^{n+1}} < \delta \quad \Rightarrow \quad \ln(b-a) - \ln 2^{n+1} < \ln \delta$$

$$\Rightarrow \quad \ln(b-a) - \ln\delta < (n+1)\ln 2$$

$$\Rightarrow \quad \frac{\ln(b-a) - \ln\delta}{\ln 2} < n+1$$

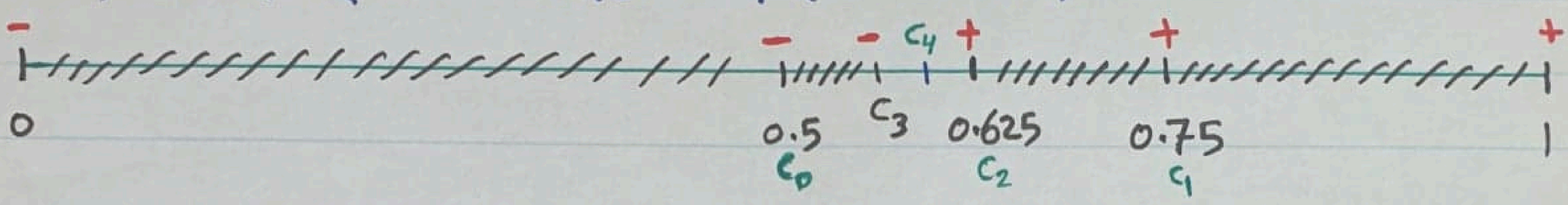
$$\Rightarrow \quad \boxed{n > \frac{\ln\left(\frac{b-a}{\delta}\right)}{\ln 2} - 1}$$

Exp Use Bisection method to find c_4 as an estimate to the root of this equation
$e^x - \cos x = 1$ on $[0,1]$

$e^x - \cos x - 1 = 0 \Rightarrow \boxed{f(x) = e^x - \cos x - 1}$ on $[0,1]$

$a_0 = 0$, $b_0 = 1$

$f(0) = -1 < 0$ and $f(1) = 1.18 > 0$



$c_0 = \frac{0+1}{2} = 0.5 \Rightarrow f(0.5) = -0.229 \Rightarrow [a_1, b_1] = [0.5, 1]$

$c_1 = \frac{0.5+1}{2} = 0.75 \Rightarrow f(0.75) = 0.385 > 0 \Rightarrow [a_2, b_2] = [0.5, 0.75]$

$c_2 = \frac{0.5+0.75}{2} = 0.625 \Rightarrow f(0.625) = 0.0573 > 0 \Rightarrow [a_3, b_3] = [0.5, 0.625]$

$c_3 = \frac{0.5+0.625}{2} = 0.5625 \Rightarrow f(0.5625) = -0.0909 < 0 \Rightarrow [a_4, b_4] = [0.5625, 0.625]$

speed slow

$c_4 = \frac{0.5625+0.625}{2} = 0.59375 \Rightarrow f(0.59375) = -0.0178$

Exp Suppose the Bisection method is used to find a zero of $f(x)$ on $[2,7]$. How many times this interval must be bisected to guarantee that the approximation c_n has an accuracy of 5×10^{-9}

$a = 2$
$b = 7$
$\delta = 5\times 10^{-9}$

$n > \frac{\ln\left(\frac{b-a}{\delta}\right)}{\ln 2} - 1 \Rightarrow n > \frac{\ln\left(\frac{\cancel{7-2}}{\cancel{5\times 10^{-9}}}\right)}{\ln 2} - 1$

$n > \frac{9\ln 10}{\ln 2} - 1 \Rightarrow n > \frac{20.72}{0.6931} - 1$

$n > 29.89 - 1 \Rightarrow n > 28.89 \Rightarrow n \in \{29, 30, 31, \dots\}$