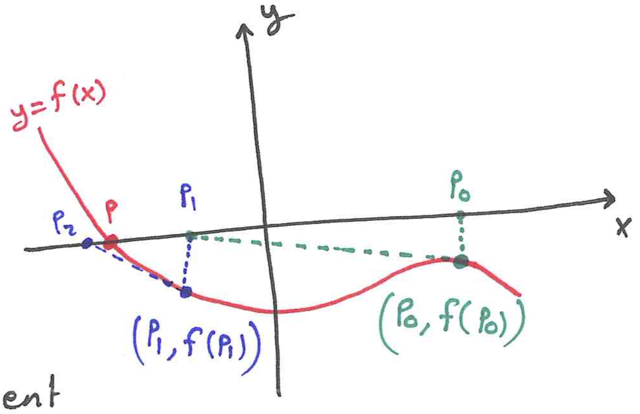
2.4 Newton – Raphson Method

- or simply " Newton's Method "
- This method develops algorithm that produces a sequence $\{p_n\}$ converges to the root p faster than Bisection and False Position methods (The best known method).
- Conditions to find the root of $f(x)=0$: $f \in C^2[a,b]$ with $f'(x) \neq 0$



- Assume the initial approximation p_0 is near the root p
- Next approximation p_1 is the point intersection between the x-axis and the line tangent to the curve at $(p_0, f(p_0))$:

$$f'(p_0) = m = \frac{0 - f(p_0)}{p_1 - p_0}$$

Hence,
$$\boxed{p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}}$$

- The process is repeated to obtain a sequence $\{p_n\}$ that converges to p. That is Newton's method iteration:

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}} \qquad n = 0, 1, 2, \ldots$$

Exp Use Newton's Method to solve $x^2 = \sin x + 1$ using $P_0 = 1.5$ with accuracy 10^{-3}.

- $f(x) = x^2 - \sin x - 1 \Rightarrow f'(x) = 2x - \cos x$

- $$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)} = P_n - \frac{P_n^2 - \sin P_n - 1}{2P_n - \cos P_n}$$

n	P_n	$\lvert P_n - P_{n-1} \rvert$
0	1.5	—
1	1.413799126	$0.0862 > 0.001$
2	1.409633752	$0.00416 > 0.001$
3	1.409624004	$0.00001 < 0.001$

Stope

180 → 3.14
85.987 → 1.5
for sin and cos

Exp Use Newton's Method with $P_0 = 1$ estimate the root of $f(x) = e^x - \cos x - 1$ with error $< 10^{-4}$

- $P_0 = 1 \Rightarrow$ $$P_{n+1} = P_n - \frac{e^{P_n} - \cos P_n - 1}{e^{P_n} + \sin P_n}$$

- $P_1 = 0.66908 3898 \Rightarrow |P_1 - P_0| > 10^{-4}$
- $P_2 = 0.603760843 \Rightarrow |P_2 - P_1| > 10^{-4}$
- $P_3 = 0.601349991 \Rightarrow |P_3 - P_2| > 10^{-4}$
- $P_4 = 0.601346767 \Rightarrow |P_4 - P_3| < 10^{-4}$ stope

180 → 3.14
57.325 → 1
for sin and cos ...

Exp Use Newton's method to estimate $\sqrt{5}$ starting with $P_0 = 2$

- Let $x = \sqrt{5} \Rightarrow x^2 = 5 \Rightarrow x^2 - 5 = 0 \Rightarrow f(x) = x^2 - 5$
  $\Rightarrow f'(x) = 2x$

- Hence, $$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$= P_n - \frac{P_n^2 - 5}{2P_n}$$

$$= \frac{P_n - \frac{5}{P_n}}{2}$$

- $P_1 = \dfrac{P_0 - \frac{5}{P_0}}{2} = \dfrac{2 + \frac{5}{2}}{2} = 2.25$

- $P_2 = \dfrac{2.25 + \frac{5}{2.25}}{2} = 2.236111111$

- $P_3 = \dfrac{2.236111111 + 5/2.236111111}{2} = 2.236067978$

- $P_4 = \dfrac{2.236067978 + 5/2.236067978}{2} = 2.236067978$

- Note that all $\{P_n\}$ with $n > 4$ will give same result as in P_4, so we see the convergence accurate to 9 decimal places.

Exp Estimate $\sqrt[3]{15}$ $\Rightarrow x = \sqrt[3]{15} \Rightarrow x^3 - 15 = 0 \Rightarrow P_{n+1} = P_n - \dfrac{P_n^3 - 15}{3P_n^2}$

- $P_0 = 2$
  $P_1 = 2.58\overline{3}$
  $P_2 = 2.471441785$
  $P_3 = 2.466223133$
  $P_4 = 2.466212074$ as in calculator

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Th (Newton-Raphson Theorem)

- Assume $f \in C^2[a,b]$ and $\exists$ a number $p \in [a,b]$ s.t $f(p)=0$.
- If $f'(p) \neq 0$, then $\exists$ $\delta > 0$ s.t the sequence $\{p_k\}_{k=0}^{\infty}$

defined by * $\boxed{p_{k+1} = g(p_k) = p_k - \dfrac{f(p_k)}{f'(p_k)} \quad \text{for } k = 0,1,2,\ldots}$

will converge to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$,

where $g(x) = x - \dfrac{f(x)}{f'(x)}$

Proof. Taylor polynomial of degree 1 about p_0 is

$$f(x) = f(p_0) + f'(p_0)(x-p_0)$$

- Substitute $x = p$ and note that $f(p) = 0 \Rightarrow$

$$0 = f(p_0) + f'(p_0)(p-p_0)$$

- Solve for $p \Rightarrow p = p_0 - \dfrac{f(p_0)}{f'(p_0)} \approx p_1$

- This is used to define the next approximation p_1 and so $*$ is established.

- To prove the convergence: Note that $g(p) = p - \dfrac{f(p)}{f'(p)} = p$ so p is fixed point of g.

- $g'(x) = 1 - \dfrac{f'f' - f f''}{(f'(x))^2} = \dfrac{f f''}{(f')^2} \Rightarrow g'(p) = 0 < 1$ and $g'(x)$ is continuous

Hence, $\exists$ $\delta > 0$ s.t $|g'(x)| < 1$ on $(p-\delta, p+\delta)$ by Th FPIT II page 21.
