Def (Multiplicity of Roots)

- Assume $f, f', \dots, f^{(M)}$ are defined and continuous on interval about the root p, where $M \in \mathbb{Z}^+$.
- We say $f(x) = 0$ has a root of order M at $x = p$ (or p has multiplicity M) iff

$$f(p) = 0, \quad f'(p) = 0, \quad f''(p) = 0, \quad \dots, \quad f^{(M-1)}(p) = 0, \quad f^{(M)}(p) \neq 0$$

Def
- A root p of order $M = 1$ is called simple root.
- A root p of order $M > 1$ is called multiple root.
  - ↳ if $M = 2$, then p is called double root.
  - ↳ if $M = 3$, then p is called cubic root.

  $\vdots$

Exp Find the roots of $f(x)$ and their multiplicity

1. $f(x) = x^3 - 3x + 2$

- one can write $f(x) = (x+2)(x-1)^2$ so $p = -2$, $p = 1$ roots
- $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$

$\boxed{p = 1}$ $\Rightarrow f(1) = 0, \ f'(1) = 0, \ f''(1) = 6 \neq 0$ so $M = 2$ and $p = 1$ is double root.

$\boxed{p = -2}$ $\Rightarrow f(-2) = 0, \ f'(-2) = 9 \neq 0$ so $M = 1$ and $p = -2$ is simple root.

$\boxed{2}$ $f(x) = (x-1)\ln x$

- $p = 1$ is the only root.
- $f'(x) = \dfrac{x-1}{x} + \ln x$ and $f''(x) = \dfrac{1}{x^2} + \dfrac{1}{x}$
- $f(1) = 0$, $f'(1) = 0$, $f''(1) = 2 \neq 0$

so The multiplicity of $p=1$ is 2 and its double root.

Lemma If $f(x) = 0$ has a root p and $\exists$ a continuous function $h(x)$ s.t

$f(x) = (x-p)^M h(x)$ where $h(p) \neq 0$ then the root p has multiplicity M

Remark • In Exp $\boxed{1}$ page 41 $\Rightarrow$ $p_1 = 1$ has $M_1 = 2$ and $p_2 = -2$ has $M_2 = 1$ so

$f(x) = (x+2)(x-1)^2$ with $h_1(x) = x+2$, $h_1(1) \neq 0$

$h_2(x) = (x-1)^2$, $h_2(-2) \neq 0$

• In Exp $\boxed{2}$ page 42 $\Rightarrow$ $p = 1$ has $M = 2$

$f(x) = (x-1)\ln x$ but $\ln x = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \cdots$

$= \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n}$ $= \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$

with $h(1) = 0$

$= (x-1)^2 - \dfrac{(x-1)^3}{2} + \dfrac{(x-1)^4}{3} + \cdots$

$\boxed{3}$ $f(x) = x^{101} - x^{100} + x^{30} - 1$ $\Rightarrow p = 1$ is root

$f'(x) = 101x^{100} - 100x^{99} + 30x^{29}$ $\Rightarrow f'(1) = 31 \neq 0$

$\Rightarrow p = 1$ has $M = 1$