Def (Speed of Convergence)

- Assume that the sequence $\{p_n\}_{n=0}^{\infty}$ converges to p.
- If there exist two positive constants $A \neq 0$ and $R > 0$ s.t

$$\lim_{n\to\infty} \frac{|p - p_{n+1}|}{|p - p_n|^R} = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = A,$$

Then we say that $\{p_n\}$ converges to p with order of convergence R.

Remarks ① We use the order of convergence R to measure the speed of convergence of any method:

- if $R = 1$, then the convergence of $\{p_n\}$ is linear
- if $R = \frac{3}{2}$, then the convergence of $\{p_n\}$ is super linear
- if $R = 2$, then the convergence of $\{p_n\}$ is quadratic
- if $R = 3$, then the convergence of $\{p_n\}$ is cubic

② when $R \uparrow \Rightarrow$ speed $\uparrow \Rightarrow$ error $\downarrow$

③ A is called the asymptotic error constant

This is because • as n gets large $\Rightarrow$

$$|E_{n+1}| \approx A\,|E_n|^R$$

- if $|E_n| = 0.01$ then for

$$R = 1 \Rightarrow |E_{n+1}| \approx A|E_n| = A\,(0.01)$$

$$R = 2 \Rightarrow |E_{n+1}| \approx A|E_n|^2 = A\,(0.01)^2 = A\,(0.0001)$$

Exp Find A and R for the following sequences:

1. $\left\{\frac{1}{10^n}\right\}_{n=0}^{\infty} = 1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$

- $\lim_{n\to\infty} \frac{1}{10^n} = 0 = p$

- $$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = \lim_{n\to\infty} \frac{|p - p_{n+1}|}{|p - p_n|^R} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{10^{n+1}}\right|}{\left|0 - \frac{1}{10^n}\right|^R}$$

$$= \lim_{n\to\infty} \frac{10^{nR}}{10^{n+1}} = \begin{cases} \boxed{\frac{1}{10} \quad \text{if } R=1} \\ \infty \quad \text{if } R>1 \quad \times \\ 0 \quad \text{if } R<1 \quad \times \end{cases}$$

$$= \frac{1}{10} \lim_{n\to\infty} 10^{n(R-1)}$$

- Hence, by definition $\Rightarrow$ $A = \frac{1}{10}$ and $R=1$

$\Rightarrow$ The convergence is linear

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2. $p_n = \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$ $\Rightarrow \lim_{n\to\infty} \frac{1}{2^n} = 0 = p$

- $$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{2^{n+1}}\right|}{\left|0 - \frac{1}{2^n}\right|^R} = \lim_{n\to\infty} \frac{2^{nR}}{2^{n+1}}$$

$$= \frac{1}{2} \lim_{n\to\infty} 2^{n(R-1)} = \begin{cases} \boxed{\frac{1}{2} \quad \text{if } R=1} \\ \infty \quad \text{if } R>1 \\ 0 \quad \text{if } R<1 \end{cases}$$

- Hence, $A = \frac{1}{2}$ and $R=1$

and the convergence is linear
