**Th** (Speed of Newton's Method)

Assume Newton-Raphson iteration

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \quad , \text{ given } p_0 \quad , \quad n = 0, 1, 2, \cdots$$

produces a sequence $\{p_n\}$ that converges to the root $p$ of the function $f(x)$. Then,

1. if $p$ is simple root $(M=1)$, then Newton's iteration $\{p_n\}$ converges to $p$ quadratically $(R=2)$ with

$$A = \left| \frac{f''(p)}{2f'(p)} \right| \quad \text{and} \quad \lim_{n \to \infty} \frac{|E_{n+1}|}{|E_n|^2} \approx A$$

2. if $p$ is a multiple root (of order $M > 1$), then Newton's iteration $\{p_n\}$ converges to $p$ linearly $(R=1)$ with

$$A = \frac{M-1}{M} \quad \text{and} \quad \lim_{n \to \infty} \frac{|E_{n+1}|}{|E_n|} \approx \frac{M-1}{M}$$

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**Exp** Let $f(x) = x^3 - 3x + 2$

1. Find the order of convergence $R$ and the asymptotic error constant $A$ when Newton-Raphson iteration is used to find the roots of $f(x) = 0$

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- Recall the roots of $f(x) = 0 \Rightarrow p = -2 \ , \ p = 1$
- Note that $p = -2$ is simple root $(M=1) \Rightarrow$ so $R = 2$ by Th above

  and hence, $A = \left| \frac{f''(-2)}{2f'(-2)} \right| = \left| \frac{-12}{2(9)} \right| = \frac{2}{3}$
- Note that $p = 1$ is multiple root $(M=2) \Rightarrow$ so $R = 1$ by Th above

  and hence, $A = \frac{M-1}{M} = \frac{2-1}{2} = \frac{1}{2}$

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[2] start with $p_0 = -2.4$ and use Newton's – Raphson iteration to find the root $p = -2$. (Prove the quadratic convergence at simple root in [1]).

$p = -2$ , $p_0 = -2.4$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{p_n^3 - 3p_n + 2}{3p_n^2 - 3}$$

$$= \frac{2p_n^3 - 2}{3p_n^2 - 3}$$

| $n$ | $p_n$ | $\lvert p_{n+1} - p_n \rvert$ (no need) | $E_n = \lvert p - p_n \rvert$ | $\lvert E_{n+1} \rvert / \lvert E_n \rvert^2$ |
|---|---|---|---|---|
| 0 | -2.4 | 0.323809524 | 0.4 | 0.476190475 |
| 1 | -2.076190476 | 0.072594465 | 0.076190476 | 0.619469086 |
| 2 | -2.003596011 | 0.003587422 | 0.003596011 | 0.664202613 $\approx \frac{2}{3}$ |
| 3 | -2.000008589 | 0.000008589 | 0.000008589 | |
| 4 | -2 | 0 | 0 | |

- Note that $|E_{n+1}| \approx A\,|E_n|^2$ for larg n
- To check this $\Rightarrow$

$|E_3| = |p - p_3| = 0.0000085\,89$

$|E_2| = |p - p_2| = 0.00359\,6011 \Rightarrow |E_2|^2 = 0.000012931$

- Now it is easy to see that

$$|E_3| \approx A\,|E_2|^2 \Leftrightarrow 0.0000085\,89 \approx \frac{2}{3}(0.000012931)$$

$$= 0.00000862\,1$$

3 Start with $P_0 = 1.2$ and use Newton's Method to prove the linear convergence at the double root $p = 1$. 47

$P = 1$ , $P_0 = 1.2$

$$P_{n+1} = \frac{2P_n^3 - 2}{3P_n^2 - 3}$$

| n | $P_n$ | $E_n = \lvert P - P_n \rvert$ | $\lvert E_{n+1} \rvert / \lvert E_n \rvert$ |
|---|---|---|---|
| 0 | 1.2 | 0.2 | 0.515151515 |
| 1 | 1.103030303 | 0.103030303 | 0.508165253 |
| 2 | 1.052356420 | 0.052356420 | 0.496751115 |
| 3 | 1.026400811 | 0.026400811 | 0.509753688 |
| 4 | 1.013257730 | 0.013257730 | 0.501097775 |
| 5 | 1.006643419 | 0.006643419 | 0.500550093 ≈ 0.5 |
| ⋮ | 1.003325375 | 0.003325375 | |
| 20 | 1.000000409 | | |

- Note that $|E_{n+1}| \approx A\,|E_n|$ for larg n

- To check this ⇒

$|E_5| = |P - P_5| = 0.006643419$

$|E_4| = |P - P_4| = 0.1325 7730$

- Now it is easy to see that

$|E_5| \approx A\,|E_4| \iff 0.006643419 \approx (0.5)(0.13257730)$

$= 0.066 28865$

**Remark.** • In the previous Exp the root $p$ was known.

• However, sometimes $p$ is unknown (see next Exp).

**Exp** (P is unknown)

Consider the equation $x^2 - \sin x - 1 = 0$

(1) Use Newton's method with $P_0 = 1.5$ to estimate the solution of this equation with error less than $10^{-3}$.

3.14 → 180
1.5 → 85.987

| n | $P_n$ | $\lvert P_{n+1} - P_n \rvert$ |
|---|---|---|
| 0 | 1.5 | |
| 1 | 1.413799126 | 0.086 |
| 2 | 1.409633752 | 0.004 |
| 3 | 1.409624004 | 0.000009748 |

$f(x) = x^2 - \sin x - 1$

$f'(x) = 2x - \cos x$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$P = 1.409624004$

(2) Find the order of convergence and the asymptotic error constant

• We find the multiplicity of the root $p$

• $f'(p) = 2(1.409624004) - \cos(80.81)$
$= 2.819248008 - 0.1597088975$
$= 2.6595391033 \neq 0$

$\Rightarrow M = 1$ and $p = 1.409624004$ is simple root.

• Hence, by Th above $R = 2$ and $A = \left| \frac{f''(p)}{2f'(p)} \right| = 0.56173286$

3 Prove part 2 Numerically

| $n$ | $P_n$ | $E_n = \|p - P_n\|$ | $\|E_{n+1}\| / \|E_n\|^2$ |
|---|---|---|---|
| 0 | 1.5 | 0.090375 | 0.5111162 |
| 1 | 1.413799126 | 0.004175 | 0.559212 $\approx A$ |
| 2 | 1.409624004 | 0.0000009748 | |

- $|E_2| = 0.0000009748$

  $|E_1| = 0.004175 \Rightarrow |E_1|^2 = 0.0000174306$

- Note that $|E_2| \approx A\,|E_1|^2 \Leftrightarrow$

$$0.0000009748 \approx (0.56173286)(0.0000174306)$$

$$= 0.00000097913$$