Proof of Th (Speed of Newton's Method)
(page 45)

Recall Newton-Raphson iteration

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \quad , \text{ given } p_0 \ , \ n = 0, 1, 2, \dots$$

Proof Part I : we need to show if p is simple root ($M=1$) then Newton's iteration $p_n \to p$ quadratically ($R=2$) with $A = \left| \frac{f''(p)}{2 f'(p)} \right|$

- Define $\boxed{g(x) = x - \frac{f(x)}{f'(x)}}$ — $*$
- since p is root of $f(x) \Rightarrow f(p) = 0 \Rightarrow$
  p is fixed point of $g \Rightarrow g(p) = p - \frac{f(p)}{f'(p)} = p$
- The Taylor expansion of $g(x)$ about the fixed point p is

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(c)}{2!}(x-p)^2 \quad , \quad c \in (x, p)$$

$$g(p_n) = g(p) + g'(p)(p_n - p) + \frac{g''(c)}{2!}(p_n - p)^2 \quad , \quad c \in (p_n, p) \quad \text{(A)}$$

- Find g' and g'' $\Rightarrow$ Use $*$

$$g'(x) = 1 - \frac{f'(x) f'(x) - f(x) f''(x)}{(f'(x))^2} = \frac{f(x) f''(x)}{[f'(x)]^2}$$

$$g''(x) = \frac{[f'(x)]^2 [f(x) f'''(x) + f'(x) f''(x)]}{[f'(x)]^4} = \frac{f(x) f'''(x) + f'(x) f''(x)}{[f'(x)]^2}$$

since p is simple root $\Rightarrow f(p) = 0$ and $f'(p) \neq 0$

- Note that $g'(p) = 0$ since $f(p) = 0$ and $f'(p) \neq 0$
  $g''(p) = \frac{f''(p)}{f'(p)} \neq 0$ since $f''(p) \neq 0$

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substitute $g'(P)=0$ and $g''(P)=\frac{f''(P)}{f'(P)}$ (B) in (A) $\Rightarrow$
$g(P)=P$

$$g(P_n) = P + 0(P_n - P) + \frac{g''(c)}{2}(P_n - P)^2$$

$$P_{n+1} = P + \frac{g''(c)}{2}(P_n - P)^2$$

$c \in (P_n, P)$

$P_n < c < P$

since $P_n \to P$ as $n \to \infty$

$$P_{n+1} - P = \frac{g''(c)}{2}(P_n - P)^2$$

$$|P_{n+1} - P| = \left|\frac{g''(c)}{2}\right| |P_n - P|^2$$

as $n \to \infty \Rightarrow c \to P$

$$\frac{|P_{n+1} - P|}{|P_n - P|^2} = \frac{1}{2}|g''(c)|$$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2} = \boxed{\frac{1}{2}\lim_{n\to\infty}|g''(c)| = \frac{1}{2}|g''(P)|}$$

using (B)

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2} = \frac{1}{2}\left|\frac{f''(P)}{f'(P)}\right|$$

Hence, $A = \left|\frac{f''(P)}{2f'(P)}\right|$ and $R = 2$

Proof Part 2: we need to show if p is multiple root $(M>1)$ then Newton Iteration converges to p linearly $(R=1)$ with $A = \frac{M-1}{M}$

- we can prove this part same way as in part A, or as follows (this method works also for $M=1$)

Since the root p is multiple $(M>1)$ $\Rightarrow$ $f(x) = (x-p)^M h(x)$
where $h(x)$ is cont. s.t $h(p) \neq 0$ (see page 42)

we know that $P_{n+1} = g(P_n) = P_n - \frac{f(P_n)}{f'(P_n)}$

Define $g(x) = x - \frac{f(x)}{f'(x)}$

$$= x - \frac{(x-p)^M h(x)}{(x-p)^M h'(x) + M(x-p)^{M-1} h(x)} \cdot \frac{(x-p)^{1-M}}{(x-p)^{1-M}}$$

$$\boxed{g(x) = x - \frac{(x-p)\,h(x)}{M\,h(x) + (x-p)\,h'(x)}}$$

Note that $g(p) = p$

- Now expand $g(x)$ about p using Taylor $\Rightarrow$

$$g(x) = g(p) + g'(c)(x-p) \qquad c \in (x, p)$$

$$g(P_n) = g(p) + g'(c)(P_n - p) \qquad c \in (P_n, p)$$

$$P_{n+1} = p + g'(c)(P_n - p)$$

$$P_{n+1} - p = g'(c)(P_n - p)$$

$$|P_{n+1} - P| = |g'(c)|\,|P_n - P| \qquad P_n < c < P$$

$$|E_{n+1}| = |g'(c)|\,|E_n|$$

$$\frac{|E_{n+1}|}{|E_n|^1} = |g'(c)| \qquad R=1$$

since $P_n \to P$ as $n \to \infty$

$\Downarrow$

$c \to P$ as $n \to \infty$

$$\lim_{n\to\infty} \left| \frac{E_{n+1}}{E_n} \right| = \boxed{\lim_{n\to\infty} |g'(c)| = |g'(P)|}$$

Hence,

$$A = |g'(P)|$$

$$Mh'(x) + (x-P)h''(x) + h'(x)$$

But
$$g'(x) = 1 - \frac{[Mh(x) + (x-P)h'(x)][(x-P)h'(x) + h(x)] - (x-P)h(x)[\quad]}{[Mh(x) + (x-P)^{M-1} h(x)]^2}$$

$$g'(P) = 1 - \frac{M h^2(P)}{M^2 h^2(P)}$$

$$= 1 - \frac{1}{M}$$

$$= \frac{M-1}{M}$$

Hence, $A = |g'(P)| = \dfrac{M-1}{M}$ since P is multiple $\Rightarrow M>1$

with $R = 1$