## 3.3 Backward Substitution

Backward substitution method used to solve a linear system of equations that has an upper-triangular coefficient matrix:

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \cdots + a_{1n} x_n &= b_1 \\
a_{22} x_2 + a_{23} x_3 + \cdots + a_{2n} x_n &= b_2 \\
a_{33} x_3 + \cdots + a_{3n} x_n &= b_3 \\
\ddots \qquad & \vdots \\
a_{nn} x_n &= b_n
\end{aligned}
$$

$$
\left[\begin{array}{ccccc|c}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\
0 & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\
0 & 0 & a_{33} & \cdots & a_{3n} & b_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn} & b_n
\end{array}\right] = [U \mid b]
$$

$U$: coefficient matrix

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**Exp** Solve the following linear system using Backward Substitution and find the cost.

$$
\begin{aligned}
4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
-2x_2 + 7x_3 - 4x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6
\end{aligned}
$$

step 1 : $x_4 = \frac{6}{3} = 2$ $\Rightarrow$ one operation

| | |
|---|---|
| $-,+$ | 0 |
| $\div, \times$ | 1 |

step 2 : $X_3 = \frac{4 - 5 \times 2}{6} = -1$ $\Rightarrow$ Three operations

| | |
|---|---|
| $-,+$ | 1 |
| $\div, \times$ | 2 |

step 3 : $X_2 = \frac{-7 \times -1 + 4 \times 2 - 7}{-2} = -4$ $\Rightarrow$ Five operations

| | |
|---|---|
| $-,+$ | 2 |
| $\div, \times$ | 3 |

step 4 : $X_1 = \frac{1 \times -4 - 2 \times -1 - 3 \times 2 + 20}{4} = 3$ $\Rightarrow$ Seven operations

| | |
|---|---|
| $-,+$ | 3 |
| $\div, \times$ | 4 |

Hence, total cost $= 16 = (4)^2 = n^2$

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Cost of Backward Substitution (B.S.) for solving $n \times n$ linear system

| step | $+,-$ | $\times, \div$ |
|---|---|---|
| 1 | 0 | 1 |
| 2 | 1 | 2 |
| 3 | 2 | 3 |
| 4 | 3 | 4 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | $n-1$ | $n$ |

Total $+,-$ is $0 + 1 + 2 + 3 + \cdots + n-1 = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$

Total $\times, \div$ is $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

Hence, total cost $= \frac{n^2-n}{2} + \frac{n^2+n}{2} = n^2$

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**Remark**

$$1+2+3+\cdots+n=\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$1+2^2+3^2+\cdots+n^2=\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

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## Five Methods to solve the linear system $AX=b$

$\boxed{1}$ Gaussian Elimination:

$$\underbrace{(A|b)}_{\text{Augmented matrix}} \longrightarrow \underbrace{(U|c)}_{\text{Upper matrix}}$$ solve by B.S.

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$\boxed{2}$ Gauss - Jordan Reduction:

$$(A|b) \longrightarrow (I|X)$$

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$\boxed{3}$ Inverse Method:

- Find $A^{-1}$:

$$(A|I) \longrightarrow (I|A^{-1})$$

- Then $X = A^{-1}b$

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$\boxed{4}$ Cramer's Rule:

- Find $|A| \neq 0$
- Then $x_1 = \frac{|A_1|}{|A|}$, $x_2 = \frac{|A_2|}{|A|}$, ..., $x_n = \frac{|A_n|}{|A|}$

## $\boxed{5}$ LU Factorization :

- Write $A = LU$ where $L$ is lower triangle matrix
  $U$ is Upper triangle matrix
- Let $Y = UX \Rightarrow$

$$AX = b \quad \text{becomes} \quad LUX = b$$
$$LY = b$$

- Now solve $LY = b$ by F.S and find $Y$
- Then solve $UX = Y$ by B.S and find $X$

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Remark: The speed of these methods is like this

$$\boxed{4} < \boxed{3} < \boxed{2} < \boxed{1} = \boxed{5}$$

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Exp show that if $A$ is $n\times n$ matrix, then the cost of finding $A^2$ is $2n^3 - n^2$

$$A^2 = AA = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

$c_{11} = a_{11} \times a_{11} + a_{12} \times a_{12} + \cdots + a_{1n} \times a_{1n}$ costs $n$ for $\times$
$n-1$ for $+$
$2n-1$

Hence, $c_{11}$ costs $2n-1$

But $A^2$ has $n^2$ elements and each one costs $2n-1$

Hence, total cost of calculating $A^2$ is

$$n^2(2n-1) = 2n^3 - n^2$$