

I Gaussian Elimination (G.E.)

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The Augmented matrix is denoted by

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array} \right] \dots *$$

Th (Elementary Row Operations ERO)

The following operations applied to the augmented matrix * yield an equivalent linear system:

Row Operation I: Interchange two rows

Row Operation II: Multiply a row by a nonzero constant

Row operation III: The row R_K can be replaced by the sum of R_K and a nonzero multiple of any other row R_P . That is, $R_K = R_K - m_{KP} R_P$

where $m_{KP} = \frac{a_{KP}}{a_{PP}}$ is called the KP multiplier

Exp Consider the following linear system:

$$x_1 + 2x_2 + x_3 + 4x_4 = 13$$

$$2x_1 \quad + 4x_3 + 3x_4 = 28$$

$$4x_1 + 2x_2 + 2x_3 + x_4 = 20$$

$$-3x_1 + x_2 + 3x_3 + 2x_4 = 6$$

$$n = 4$$

i Express this system in augmented matrix form.

Pivot

$$m_{21} = \frac{a_{21}}{a_{11}} = 2$$

$$m_{31} = \frac{a_{31}}{a_{11}} = 4$$

$$m_{41} = \frac{a_{41}}{a_{11}} = -3$$

are the multipliers

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array} \right) = (A | b)$$

Pivot row

or $(12)(2) = 24 + 3$
 entry operations
 $+ : - : 1$
 $\times : 1$

multiplies

ii Find an equivalent upper-triangular system using G.E.

• Step 1 • Find multipliers m_{21}, m_{31}, m_{41}

Apply them

$$R_2 - 2R_1$$

$$R_3 - 4R_1$$

$$R_4 - 3R_1$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & -6 & -2 & -15 & -32 \\ 0 & 7 & 6 & 14 & 45 \end{array} \right)$$

Costs

$$\div : 3$$

$$+ (- : 3(4))$$

$$\times : 3(4)$$

• Step 2 • Find multipliers $m_{32} = \frac{-6}{-4} = 1.5$ and $m_{42} = \frac{7}{-4} = -1.75$

Apply them

$$R_3 - 1.5R_2$$

$$R_4 + 1.75R_2$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 9.5 & 5.25 & 48.5 \end{array} \right)$$

$$\div : 2$$

$$+ (- : 2(3))$$

$$\times : 2(3)$$

• step 3 • Find the multiplier $m_{43} = \frac{9.5}{-5} = -1.9$

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• Apply it

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array} \right)$$

$$\begin{array}{l} \div : 1 \\ +, - : 1(2) \\ \times : 1(2) \end{array}$$

iii Use B.S to solve the resulting system

$$x_4 = \frac{-18}{-9} = 2$$

$$x_3 = \frac{7.5(2) - 35}{-5} = 4$$

$$x_2 = \frac{-2(4) + 5(2) + 2}{-4} = -1$$

$$x_1 = 13 - 2(-1) - 4 - 4(2) = 3$$

iv Find the cost of solving this system using G.E.

• Cost 1: $(A|b) \rightarrow (U|c)$ for this 4×4 system

Step	+, -	÷, ×
1	3(4)	3, 3(4)
2	2(3)	2, 2(3)
3	1(2)	1, 1(2)
Total	20	26

cost 1 = 46

• Cost 2: Using B.S. \Rightarrow cost 2 = $n^2 = 4^2 = 16$

Hence, total cost for 4×4 system = cost 1 + cost 2 = 62

Result ① The cost of reducing $n \times n$ linear system 62
to an upper-triangular system using G.E. is

$$\frac{4n^3 + 3n^2 - 7n}{6}$$

② The cost of solving $n \times n$ linear system using G.E. is

$$\frac{4n^3 + 9n^2 - 7n}{6}$$

Proof

① Cost of $(A|b) \rightarrow (U|c)$ as we have seen in Exp - iv page 61 is as follow:

step	+, -	÷	x
1	$(n-1)n$	$n-1$	$(n-1)n$
2	$(n-2)(n-1)$	$n-2$	$(n-2)(n-1)$
⋮	⋮	⋮	⋮
k	$(n-k)(n-k+1)$	$n-k$	$(n-k)(n-k+1)$
⋮	⋮	⋮	⋮
Total	$\sum_{k=1}^{n-1} (n-k)(n-k+1)$	$\sum_{k=1}^{n-1} (n-k)$	$\sum_{k=1}^{n-1} (n-k)(n-k+1)$

Hence, total cost is $\sum_{k=1}^{n-1} [2(n-k)(n-k+1) + n-k]$

$$= \sum_{m=1}^{n-1} [2m(m+1) + m] = \sum_{m=1}^{n-1} 2m^2 + 3m$$

$m = n - k$

$k=1 \Rightarrow m = n-1$

$k=n-1 \Rightarrow m=1$

$$= 2 \frac{n(n-1)(2n-1)}{6} + 3 \frac{n(n-1)}{2} = \frac{4n^3 + 3n^2 - 7n}{6}$$

② Cost of any $n \times n$ system using G.E. is cost of $(A|b) \rightarrow (U|c)$ + cost of B.S.

$$= \frac{4n^3 + 3n^2 - 7n}{6} + n^2 = \frac{4n^3 + 9n^2 - 7n}{6}$$

Gaussian Elimination and Pivoting

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- If the pivot element $a_{pp} = 0$ in row p , then row p can not be used to eliminate the elements in column p below the main diagonal.
- The process of finding a row k with nonzero pivot element $a_{kp} \neq 0$, $k > p$ and interchanging row p by row k is called **pivoting**.
- **Pivoting** is two types:

□ Trivial pivoting:

- if $a_{pp} \neq 0$, then do not switch rows
- if $a_{pp} = 0$, then switch row p by the first row k below p s.t. $a_{kp} \neq 0$.

Exp

$$\begin{bmatrix} 0 & -1 & 2 & | & 3 \\ 4 & 3 & -1 & | & 2 \\ 7 & 0 & -3 & | & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 3 & -1 & | & 2 \\ 0 & -1 & 2 & | & 3 \\ 7 & 0 & -3 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 2 & | & 3 \\ 0 & 2 & 4 & | & 7 \\ 1 & 2 & 3 & | & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 2 & 4 & | & 7 \\ 0 & -1 & 2 & | & 3 \end{bmatrix}$$

2 Partial Pivoting:

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- if $a_{pp} = 0$ or $a_{pp} \neq 0$, choose the pivotal row k whose pivot a_{kp} , $k > p$ satisfy $|a_{kp}| = \max\{|a_{pp}|, |a_{p+1,p}|, \dots, |a_{np}|\}$
- This will make all multipliers m_{rp} , $r = p+1, \dots, n$ less than or equal to 1 in absolute value.
- And hence, reducing the error being propagated when using a finite-digit arithmetic

Exp Consider the following linear system

$$\begin{aligned} 1.133 x_1 + 5.281 x_2 &= 6.414 \\ 24.14 x_1 - 1.210 x_2 &= 22.93 \end{aligned}$$

whose solution is $(x_1, x_2) = (1, 1)$.

1 Use G.E. with **trivial pivoting** and use four-digit arithmetic to solve this system.

pivot \rightarrow $\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & -1.210 & 22.93 \end{array} \right]$ since pivot $a_{11} \neq 0$
so we do not switch rows

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{24.14}{1.133} = 21.31$$

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array} \right]$$

$$x_2 = \frac{-113.8}{-113.7} = 1.001$$

$$x_1 = \frac{6.414 - 5.281(1.001)}{1.133} = 0.9956$$

Note that the error in the solution is due to the magnitude of the multiplier m_{21} which is $\gg 1$.

2] Use G.E. with **partial pivoting** and use four-digit arithmetic to solve this system.

pivot $\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 1.133 & 5.281 & 6.414 \end{array} \right]$

- $\max\{1.133, 24.14\} = 24.14$
- Hence, pivot = 24.14

$$m_{21} = \frac{1.133}{24.14} = 0.04693 < 1$$

$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array} \right]$$

$$x_2 = \frac{5.338}{5.338} = 1$$

$$x_1 = \frac{22.93 + 1.210(1)}{24.14} = 1$$

No Error

