2 LU - Factorization

To solve the linear system $AX = b$:

- Write $A = L\,U$ where L is lower triangular matrix and U is upper triangular matrix

- Let $Y = U\,X \Rightarrow$

  $AX = b$ becomes $L\,U\,X = b$

  $$L\,Y = b$$

- Now solve $LY = b$ by F.S and find Y

- Then solve $U\,X = Y$ by B.S and find X

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* We will see that :

cost of LU = Cost of G.E.

Exp ① Use LU factorization to solve the linear system:

$$4x_1 + 3x_2 - x_3 = 1$$
$$-2x_1 - 4x_2 + 5x_3 = 6$$
$$x_1 + 2x_2 + 6x_3 = 14$$

② find the cost of this method

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①
$$A = \begin{pmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 6 \\ 14 \end{pmatrix}$$

- We need to write $A = LU$
- First we find U using row operations:

step 1
$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{-2}{4} = -0.5 \Rightarrow R_2 + 0.5R_1$$

$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{1}{4} = 0.25 \Rightarrow R_3 - 0.25R_1$$

$$\begin{pmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 1.25 & 6.25 \end{pmatrix}$$

cost
$\div$: 2
$+,-$: 2(2)
$\times$: 2(2)

step 2
$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{1.25}{-2.5} = -0.5 \Rightarrow R_3 + 0.5R_2$$

$$U = \begin{pmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{pmatrix}$$

cost
$\div$: 1
$+,-$: 1(1)
$\times$: 1(1)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 1 \end{pmatrix}$$

Note that cost of $A = LU$ is $13 = \left.\dfrac{4n^3 - 3n^2 - n}{6}\right|_{n=3}$

Now solve $LY = b$ using F.S. where $Y = UX$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -0.5 & 1 & 0 & 6 \\ 0.25 & -0.5 & 1 & 14 \end{array}\right)$$

$y_1 = 1$

$y_2 = 6 + 0.5(1) = 6.5$

$y_3 = 14 - 0.25(1) + 0.5(6.5) = 17$

$\underline{\text{cost}}$

$\div : 0$

$+,- : 3$

$\times : 3$

Note that cost of F.S. is $n^2 - n = 3^2 - 3 = 9 - 3 = 6$

Now solve $UX = Y$ using B.S.

$$\left(\begin{array}{ccc|c} 4 & 3 & -1 & 1 \\ 0 & -2.5 & 4.5 & 6.5 \\ 0 & 0 & 8.5 & 17 \end{array}\right)$$

$x_3 = \dfrac{17}{8.5} = 2$

$x_2 = \dfrac{6.5 - 4.5(2)}{-2.5} = \dfrac{-2.5}{-2.5} = 1$

$x_1 = \dfrac{1 + (2)(+1) - 3(1)}{4} = \dfrac{0}{4} = 0$

$\underline{\text{cost}}$

$\div : 3$

$+,- : 3$

$\times : 3$

Note that cost of B.S. is $n^2 = 3^2 = 9$

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② Hence, total cost is $13+6+9=28$

Remark The total cost of solving any linear $n\times n$ system $AX=b$ using LU factorization is cost of LU + cost of F.S. + cost of B.S.

$$= \frac{4n^3-3n^2-n}{6} + n^2-n + n^2$$

$$= \frac{4n^3+9n^2-7n}{6}$$

= cost of G.E.

Exp Let A be 3×3 matrix and b_1, b_2 are 3×1 vectors. Find the cost of solving the linear systems $AX=b_1$ and $AX=b_2$ using LU factorization.

cost of $A=LU$ is $\left.\frac{4n^3-3n^2-n}{6}\right|_{n=3} = 13$

cost of solving $Ax=b_1 \Rightarrow LY=b_1$ costs $\left. n^2-n\right|_{n=3} = 6$ by FS

$UX=Y$ costs $\left. n^2\right|_{n=3} = 9$ by BS

cost of solving $Ax=b_2 \Rightarrow LY=b_2$ costs 6 using FS

$UX=Y$ costs 9 using BS

Total cost $= 13+6+9+6+9$

$= 43$