$\boxed{3}$ Cramer's Rule

To solve the linear $n \times n$ system $AX = b$:

- Find $|A|$ « it should not be zero »
- The solution $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is obtained as follow:

$$x_i = \frac{|A_i|}{|A|}$$

where A_i is obtained by replacing the i^{th} column of A by the column b for all $i = 1, 2, \dots, n$

Exp Solve the following linear system using Cramer's Rule

$$2x_1 - 3x_2 = 8$$
$$x_1 + 5x_2 = -9$$

Then find the cost.

- $A = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix} \Rightarrow |A| = (2)(5) - (1)(-3) = 13$ with cost:

$+,-$: 1
$\times$: 2
$\overline{\ 3}$

- $x_1 = \dfrac{|A_1|}{|A|} = \dfrac{\begin{vmatrix} 8 & -3 \\ -9 & 5 \end{vmatrix}}{13} = \dfrac{13}{13} = 1$ with cost $= 3 + 1 = 4$

$x_2 = \dfrac{|A_2|}{|A|} = \dfrac{\begin{vmatrix} 2 & 8 \\ 1 & -9 \end{vmatrix}}{13} = \dfrac{-26}{13} = -2$ with cost $= 3 + 1 = 4$

Total cost $= 3\,(3) + 2 = 11$

number of determinants

cost of each one

division

Exp Assume $Ax = b$ is 3×3 linear system.
Find the cost of solving this system using Cramer's Rule.

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad x_3 = \frac{|A_3|}{|A|}$$

$$\text{Total cost} = 4\,(14) + 3 = 59$$

number of determinants

cost of each one

number of divisions

Remark To solve $n\times n$ linear system by Cramer's Rule, the cost will be

$$(n+1)\,D_n + n$$

where D_n is the cost of $|A|$

Exp : For 4×4 system $\Rightarrow$ the cost is

- $5\,D_4 + 4$ where

D_4 is the cost of $|A| = 4\,D_3 + \overset{\times}{4} + \overset{+}{3}$

$$= 4\,(14) + 7$$

$$= 63$$

- Hence, $\text{cost} = 5\,D_4 + 4$

$$= 5\,(63) + 4$$

$$= 319$$