

Solving Nonlinear Systems of Equations

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We will study the following three Methods:

[1] Newton's Method \rightarrow For 2×2 nonlinear system

[2] Fixed Point Iteration

[3] Gauss-Seidel Iteration

\rightarrow For 2×2 or 3×3 nonlinear systems

[1] Newton's Method

• Given 2×2 non linear system

$$f(x, y) = 0$$

$$g(x, y) = 0$$

with initial point (x_0, y_0)

• This method find a sequence of points

$(x_1, y_1), (x_2, y_2), \dots$ that approximates (x, y)

• We will only find (x_1, y_1) as follow:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \bar{J}^{-1}(x_0, y_0) \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix} \quad \dots *$$

where \bar{J} is the Jacobian matrix given by

$$\bar{J}(x, y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

Exp Solve the following nonlinear system using Newton's method with initial guess $(1, \frac{1}{2})$

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$$x^2 + y^2 = 10$$

$$xy = 5$$

use three significant digits and find the first iteration

$$\bullet \quad x^2 + y^2 - 10 = 0 \quad \rightarrow \quad f(x, y) = x^2 + y^2 - 10$$

$$xy - 5 = 0 \quad \rightarrow \quad g(x, y) = xy - 5$$

$$\bullet \quad (x_0, y_0) = (1, \frac{1}{2}) \quad \rightarrow \quad f(1, \frac{1}{2}) = 1 + 0.25 - 10 = -8.75$$

$$g(1, \frac{1}{2}) = 1 - 5 = -4.50$$

$$\bullet \quad \overline{J}(x, y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix} \quad \rightarrow \quad \overline{J}(1, \frac{1}{2}) = \begin{pmatrix} 2 & 1 \\ 0.5 & 1 \end{pmatrix}$$

$$\bullet \quad \overline{J}^{-1}(1, \frac{1}{2}) = \frac{1}{1.5} \begin{pmatrix} 1 & -1 \\ -0.5 & 2 \end{pmatrix} = \begin{pmatrix} 0.667 & -0.667 \\ -0.334 & 1.33 \end{pmatrix}$$

$$\bullet \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \overline{J}^{-1}(x_0, y_0) \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} - \begin{pmatrix} 0.667 & -0.667 \\ -0.334 & 1.33 \end{pmatrix} \begin{pmatrix} -8.75 \\ -4.50 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} - \begin{pmatrix} -2.84 \\ -3.07 \end{pmatrix} = \begin{pmatrix} 3.84 \\ 3.57 \end{pmatrix}$$

$$\text{10 find } \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \overline{J}^{-1}(x_1, y_1) \begin{pmatrix} f(x_1, y_1) \\ g(x_1, y_1) \end{pmatrix}$$

Exp Find the first iteration that approximates the solution of the following nonlinear system

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$$x^2 - 2x - y + 0.5 = 0$$

$$x^2 + 4y^2 - 4 = 0$$

using Newton's method and starting with $(2, 0.25)$.

$$\bullet f(x, y) = x^2 - 2x - y + 0.5 \Rightarrow f(2, 0.25) = 0.25$$

$$g(x, y) = x^2 + 4y^2 - 4 \Rightarrow g(2, 0.25) = 0.25$$

$$\bullet (x_0, y_0) = (2, 0.25)$$

$$\bullet J(x, y) = \begin{pmatrix} 2x-2 & -1 \\ 2x & 8y \end{pmatrix} \Rightarrow J(2, 0.25) = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}$$

$$\bullet J^{-1}(2, 0.25) = \frac{1}{8} \begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.125 \\ -0.5 & 0.25 \end{pmatrix}$$

$$\bullet \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 0.25 & 0.125 \\ -0.5 & 0.25 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 0.0938 \\ -0.0625 \end{pmatrix}$$

$$= \begin{pmatrix} 1.91 \\ 0.313 \end{pmatrix}$$

Exp show the formula * page 76

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- Given the following nonlinear system

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0 \end{aligned} \quad \text{starting at } (x_0, y_0)$$

- The Taylor series expansions of f and g at (x_0, y_0) are

$$f(x, y) \approx f(x_0, y_0) + f_x^{(x_0, y_0)}(x - x_0) + f_y^{(x_0, y_0)}(y - y_0)$$

$$g(x, y) \approx g(x_0, y_0) + g_x^{(x_0, y_0)}(x - x_0) + g_y^{(x_0, y_0)}(y - y_0)$$

- But $f(x, y) = g(x, y) = 0 \Rightarrow$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix} + \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

- Multiply both sides by $J^{-1}(x_0, y_0) \Rightarrow$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = J^{-1} \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

- Rearrange the last equation \Rightarrow

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J^{-1}(x_0, y_0) \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix}$$