2 Fixed Point Iteration (F.P.I)

- This method can be used to solve 2x2 or 3x3 nonlinear systems:

$$f_1(x,y)=0 \qquad\qquad f_1(x,y,z)=0$$
$$f_2(x,y)=0 \qquad \text{or} \qquad f_2(x,y,z)=0$$
$$f_3(x,y,z)=0$$

- For 2x2 system:

→ write
$$x = g_1(x,y)$$
$$y = g_2(x,y) \qquad (1)$$

→ Given $(x_0, y_0) = (p_0, q_0)$

→ The FPI is

$$p_{n+1} = g_1(p_n, q_n)$$
$$q_{n+1} = g_2(p_n, q_n)$$

$n = 0, 1, 2, \dots$

- For 3x3 system:

→ write
$$x = g_1(x,y,z)$$
$$y = g_2(x,y,z) \qquad (2)$$
$$z = g_3(x,y,z)$$

→ Given $(x_0, y_0, z_0) = (p_0, q_0, r_0)$

→ The FPI is

$$p_{n+1} = g_1(p_n, q_n, r_n)$$
$$q_{n+1} = g_2(p_n, q_n, r_n)$$
$$r_{n+1} = g_3(p_n, q_n, r_n)$$

Def
- The point (p, q) is fixed point of the system (1) if

$$p = g_1(p,q) \quad \text{and}$$
$$q = g_2(p,q).$$

- The point (p,q,r) is fixed point of the system (2) if

$$p = g_1(p,q,r) \quad \text{and}$$
$$q = g_2(p,q,r) \quad \text{and}$$
$$r = g_3(p,q,r).$$

Exp Find the fixed points of the following system

$$x - \sin y = 0$$

$$x^2 + \cos^2 y = \frac{y}{\pi} + \frac{1}{2}$$

- $x = g_1(x,y) \Leftrightarrow x = \sin y$

  $y = g_2(x,y) \Leftrightarrow y = \left(x^2 + \cos^2 y - \frac{1}{2}\right)\pi$

- $y = \left(\sin^2 y + \cos^2 y - \frac{1}{2}\right)\pi = \left(1 - \frac{1}{2}\right)\pi = \frac{\pi}{2}$

  $x = \sin y = \sin\frac{\pi}{2} = 1$

  Hence, $(p, q) = (x, y) = (1, \frac{\pi}{2})$ is fixed point

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Exp* Consider the following nonlinear system:

$$x^2 + y^2 - x = 0$$

$$e^x + y^2 - y = 0$$

Use initial approximation $(p_0, q_0) = (0.5, 0.4)$ to find the next three approximation using the FPI. 3-digits

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- $x = g_1(x,y) = x^2 + y^2 \Rightarrow p_{n+1} = p_n^2 + q_n^2$

  $y = g_2(x,y) = e^x + y^2 \Rightarrow q_{n+1} = e^{p_n} + q_n^2$

- $p_1 = g_1(p_0, q_0) = g_1(0.5, 0.4) = 0.25 + 0.16 = 0.41$

  $q_1 = g_2(p_0, q_0) = g_2(0.5, 0.4) = 1.65 + 0.16 = 1.81$

- $p_2 = g_1(p_1, q_1) = g_1(0.41, 1.81) = 0.168 + 3.28 = 3.45$

  $q_2 = g_2(p_1, q_1) = g_2(0.41, 1.81) = 1.51 + 3.28 = 4.79$

- $p_3 = g_1(p_2, q_2) = g_1(3.45, 4.79) = 11.9 + 22.9 = 34.8$

  $q_3 = g_2(p_2, q_2) = g_2(3.45, 4.79) = 31.5 + 22.9 = 54.4$

- Note that the FPI here diverges (see $\boxed{2}$ in Remark below).

Th* (Convergence of FPI - Two dimensions)

- Assume (p, q) is fixed point of $x = g_1(x, y)$ and $y = g_2(x, y)$.
- If (p_0, q_0) is sufficiently close to (p, q) and if

$$\left|\frac{\partial g_1}{\partial x}(p, q)\right| + \left|\frac{\partial g_1}{\partial y}(p, q)\right| < 1 \qquad \text{and}$$

$$\left|\frac{\partial g_2}{\partial x}(p, q)\right| + \left|\frac{\partial g_2}{\partial y}(p, q)\right| < 1$$

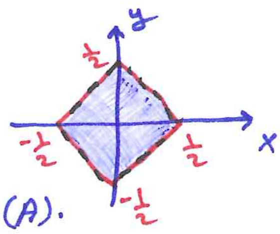
then the FPI converges to the fixed point (p, q)

Remarks: $\boxed{1}$ Converges of FPI for three dimensions follows similarly to Th above by adding z-component.

$\boxed{2}$ In Exp* page 81 $\Rightarrow$ note that

(A) ... $\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = 2|x| + 2|y| < 1 \Leftrightarrow |x| + |y| < \frac{1}{2}$

(B) ... $\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = e^x + 2|y| < 1$



but $(p_0, q_0) = (0.5, 0.4)$ does not satisfy (A).
That is why the FPI in this Exp* diverges from the fixed point.

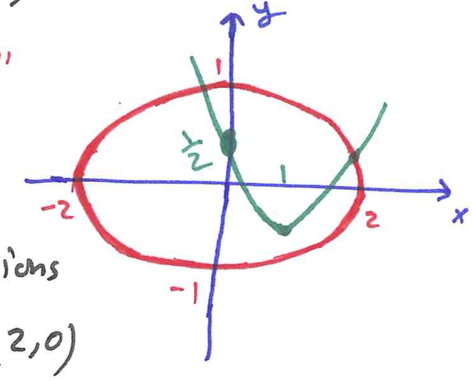
Exercise $\boxed{1}$ Find the fixed point in Exp*

$\boxed{2}$ Solve Exp* again using $(p_0, q_0) = (-0.45, 0.04)$ and show the FPI still diverges.

Exp Consider the following nonlinear system:

$$y = x^2 - 2x + 0.5 \quad \text{“parabola”}$$

$$x^2 + 4y^2 = 4 \quad \text{“Ellipse”}$$



Use the FPI to approximate the solutions using [1] $(p_0, q_0) = (0,1)$ [2] $(p_0, q_0) = (2,0)$

- The system is equivalent to

$$x = g_1(x,y) = \frac{x^2 - y + 0.5}{2}$$

$* \cdots$

$$y = g_2(x,y) = \frac{-x^2 - 4y^2 + 8y + 4}{8}$$

add to each side $-8y$

- This system has two solutions (or fixed points of $*$):

$$(p,q) \in \{(-0.2, 1), (1.9, 0.3)\}$$

- To find the first solution $(p,q) = (-0.2, 1)$ we apply formula $*$ as follows:

$$p_{n+1} = \frac{p_n^2 - q_n + 0.5}{2} = g_1(p_n, q_n)$$

$$\cdots (1)$$

$$q_{n+1} = \frac{-p_n^2 - 4q_n^2 + 8q_n + 4}{8} = g_2(p_n, q_n)$$

① $(p_0, q_0) = (0, 1)$

n	p_n	q_n
1	-0.25	1
2	-0.21875	0.9921875
3	-0.2221680	0.9939880
4	-0.2223147	0.9938121
5	-0.2221941	0.9938029
6	-0.2222163	0.9938095

This FPI converges to the first solution

② $(p_0, q_0) = (2, 0)$

n	p_n	q_n
1	2.25	0
2	2.78125	-0.1328125
3	4.184082	-0.6085510
4	9.307547	-2.4820360
5	44.80623	-15.891091
6	1 011.995	-392.60426

This FPI diverges

- Note that Theorem* in page 82 can be used to show that iteration (1) converges to the fixed point near $(-0.2, 1)$:

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |x| + 0.5 < 1 \quad \Leftrightarrow \quad |x| < 0.5$$

(2)

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = \frac{|x|}{4} + |1-y| < \frac{0.5}{4} + |1-y| < 1 \quad \Leftrightarrow$$

$$0.125 < y < 1.875$$

- The fixed point $(p, q) = (-0.2, 1)$ satisfy (2) and so Th implies that the FPI converges to $(p, q) = (-0.2, 1)$.

- However, the fixed point $(p, q) = (1.9, 0.3)$ does not satisfy (2):

$$\left|\frac{\partial g_1}{\partial x}(1.9, 0.3)\right| + \left|\frac{\partial g_1}{\partial y}(1.9, 0.3)\right| = 2.4 > 1$$

$$\left|\frac{\partial g_2}{\partial x}(1.9, 0.3)\right| + \left|\frac{\partial g_2}{\partial y}(1.9, 0.3)\right| = 1.16 > 1$$

So the FPI diverges from $(1.9, 0.3)$ if we use (1).

- Hence, the iteration (1) can not be used to find the second solution $(1.9, 0.3)$.

- To find this solution, we need a different formula for this iteration (1).
- If we add $-2x$ to the first equation and $-11y$ to the second equation, we get

$$x^2 - 4x - y + 0.5 = -2x$$
$$x^2 + 4y^2 - 11y - 4 = -11y$$

- The iteration now is

$$p_{n+1} = g_1(p_n, q_n) = \frac{-p_n^2 + 4p_n + q_n - 0.5}{2}$$
$$q_{n+1} = g_2(p_n, q_n) = \frac{-p_n^2 - 4q_n^2 + 11q_n + 4}{11} \quad \cdots (2)$$

- starting from same point $(p_0, q_0) = (2, 0) \Rightarrow$

n	p_n	q_n
1	1.75	0
2	1.71875	0.0852273
3	1.753063	0.1776676
4	1.808345	0.2504410
8	1.903595	0.3160782
12	1.900924	0.3112267
16	1.900652	0.3111994
20	1.900677	0.3112196

The FPI converges to the second solution using formula (2)