Linearization

Exp
- Given the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.
- Find the least-squar exponential curve of the form

$$y = c\, e^{Dx}$$

- $$E(C, D) = \sum_{i=1}^{n} \left(C e^{Dx_i} - y_i \right)^2$$

- $$\frac{\partial E}{\partial C} = 2 \sum_{i=1}^{n} \left(\left(C e^{Dx_i} - y_i \right) e^{Dx_i} \right) = 0 \iff$$

$$\boxed{C \sum_{i=1}^{n} e^{2Dx_i} - \sum_{i=1}^{n} y_i\, e^{Dx_i} = 0} \text{ — (1)}$$

- $$\frac{\partial E}{\partial D} = 2 \sum_{i=1}^{n} \left(C e^{Dx_i} - y_i \right) \cancel{C}\, x_i\, e^{Dx_i} = 0 \iff$$

$$\boxed{C \sum_{i=1}^{n} x_i\, e^{2Dx_i} - \sum_{i=1}^{n} y_i\, x_i\, e^{Dx_i} = 0} \text{ — (2)}$$

- The normal equations (1) and (2) are hard to solve and find C and D.
- So we use a technique called linearization.

- linearization for $y = c e^{Dx}$ works like this:
  - Take logarithm of both sides:

$$\ln y = Dx + \ln c$$

  - Then introduce the change of variables:

$$Y = Dx + E \quad \text{where } Y = \ln y, \; E = \ln C$$

  - Now use the linear normal equations page 109

$$D \sum_{i=1}^{n} x_i^2 + E \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i Y_i$$

$$D \sum_{i=1}^{n} x_i + nE = \sum_{i=1}^{n} Y_i \qquad *$$

- Solve these equations for D and E $\Rightarrow$

Then $C = e^{E}$ and so $y = f(x) = C e^{Dx}$

Exp Find the exponential fit $y = C e^{Dx}$ using linearization for the following five data points:

$(0, 1.5), (1, 2.5), (2, 3.5), (3, 5), (4, 7.5)$

- First we solve the linear normal equations $*$ and find the constants D and E
- Then we find $C = e^{E}$
- Hence, $y = f(x) = C e^{Dx}$

x_i	y_i	x_i^2	$Y_i = \ln y_i$	$x_i Y_i$
0	1.5	0	0.405465	0
1	2.5	1	0.916291	0.916291
2	3.5	4	1.252763	2.505526
3	5	9	1.609438	4.828314
4	7.5	16	2.014903	8.059612
Total		30	6.198860	16.309743

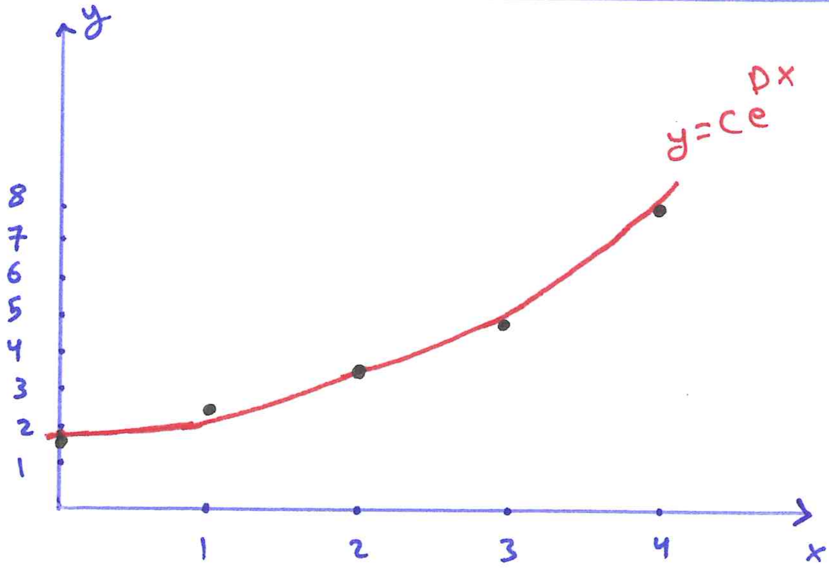
- The linear normal equations become:

$$30\,D + 10\,E = 16.309743$$
$$10\,D + 5\,E = 6.198860$$

- The solution is $D = 0.3912023$ and $E = 0.457367$
- Now we find $C = e^{E} = e^{0.457367} = 1.579910$
- Hence, $y = f(x) = C\,e^{Dx}$

$$= 1.579910\,e^{0.3912023\,x}$$

The exponential fit $y = 1.579910\,e^{0.3912023x}$ obtained by using the linearization method.



Exp Given the following data

x	1	2	4	5
y	2	8	4	6

Use two different linearization to find the fit of the form $g(x) = \frac{Cx}{D+x}$. Then estimate y when $x = 3$.

1^{st} linearization $\quad y = \frac{Cx}{D+x} \Rightarrow \frac{1}{y} = \frac{D+x}{Cx} = \frac{D}{C}\frac{1}{x} + \frac{1}{C}$

• Let $Y = \frac{1}{y}$, $X = \frac{1}{x}$

$$\boxed{Y = \alpha X + \beta}$$

where $\alpha = \frac{D}{C}$, $\beta = \frac{1}{C}$

• Now solve normal equations:

$\alpha \sum X_i^2 + \beta \sum X_i = \sum X_i Y_i \Rightarrow 1.3525\,\alpha + 1.95\,\beta = 0.6584$

$\alpha \sum X_i + n\beta = \sum Y_i \Rightarrow 1.95\,\alpha + 4\,\beta = 1.0417$

$\Downarrow$

x	y	$X_i = \frac{1}{x_i}$	$Y_i = \frac{1}{y_i}$	X_i^2	$X_i Y_i$
1	2	1	0.5	1	0.5
2	8	0.5	0.125	0.25	0.0625
4	4	0.25	0.25	0.0625	0.0625
5	6	0.2	0.1667	0.04	0.0334
		1.95	1.0417	1.3525	0.6584

$\alpha = 0.3767$

$\beta = 0.07777$

$C = \frac{1}{\beta} = 12.86$

$D = \alpha C = 4.844$

$$g(x) = \frac{Cx}{D+x} = \frac{12.86\,x}{4.844 + x}$$

when $x = 3 \Rightarrow y(3) \approx g(3) = \frac{(12.86)(3)}{4.844+3} \approx 4.918$

2^{nd} linearization

$$y = \frac{Cx}{D+x} \Rightarrow \frac{y}{x} = \frac{C}{D+x} \Rightarrow \frac{x}{y} = \frac{D+x}{C}$$

$$\frac{x}{y} = \frac{D}{C} + \frac{1}{C}x$$

- Let $Y = \frac{x}{y}$, $A = \frac{1}{C}$, $B = \frac{D}{C}$, $\boxed{Y = Ax + B}$
- Now solve the normal equations:

$$A\sum x_i^2 + B\sum x_i = \sum x_i Y_i \Rightarrow 46A + 12B = 9.165$$
$$A\sum x_i + Bn = \sum Y_i \Rightarrow 12A + 4B = 2.583$$

$$\Downarrow$$

x	y	$Y_i = \frac{x_i}{y_i}$	x_i^2	$x_i Y_i$
1	2	0.5	1	0.5
2	8	0.25	4	0.5
4	4	1	16	4
5	6	0.833	25	4.165
12		2.583	46	9.165

$A = 0.1416$
$B = 0.221$

$$C = \frac{1}{A} = 7.06$$

$$D = BC = 1.56$$

$$g(x) = \frac{Cx}{D+x} = \frac{7.06x}{1.56+x}$$

when $x = 3 \Rightarrow y(3) \approx g(3) = \frac{(7.06)(3)}{1.56+3} \simeq 4.645$

Exp Given the data $(0,1), (1,2), (3,4), (5,3)$. Use linearization to find the best fitting curve of the form $y = A x^{\beta}$ through these points.

- $y = A x^{\beta} \Rightarrow \ln y = \ln(A x^{\beta}) = \ln A + \beta \ln x$
- Let $\bar{Y} = \ln y$ and $\bar{X} = \ln x$ and $\alpha = \ln A$

$$\bar{Y} = \alpha + \beta \bar{X}$$

- The normal linear equations are:

$$\beta \sum \bar{X}_i^2 + \alpha \sum \bar{X}_i = \sum \bar{X}_i \bar{Y}_i$$

$$\beta \sum \bar{X}_i + \alpha n = \sum \bar{Y}_i$$

x_i	y_i	$\bar{Y}_i = \ln y_i$	$\bar{X}_i = \ln x_i$	$\bar{X}_i^2$	$\bar{X}_i \bar{Y}_i$
~~0~~	~~1~~		~~undefined~~		
1	2	0.6931	0	0	0
3	4	1.386	1.099	1.208	1.523
5	3	1.099	1.609	2.589	1.768
		3.178	2.708	3.797	3.291

we ignore the point $(0,1)$ $\Downarrow$ $n = 3$

$$\left.\begin{aligned} 3.797\,\beta + 2.708\,\alpha &= 3.291 \\ 2.708\,\beta + 3\,\alpha &= 3.178 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= 0.7775 \\ \beta &= 0.3122 \end{aligned}$$

But $\alpha = \ln A \Rightarrow A = e^{\alpha} = (2.178)^{0.7775} = 1.832$

Hence, $y = A x^{\beta} = 1.832\, x^{0.3122}$