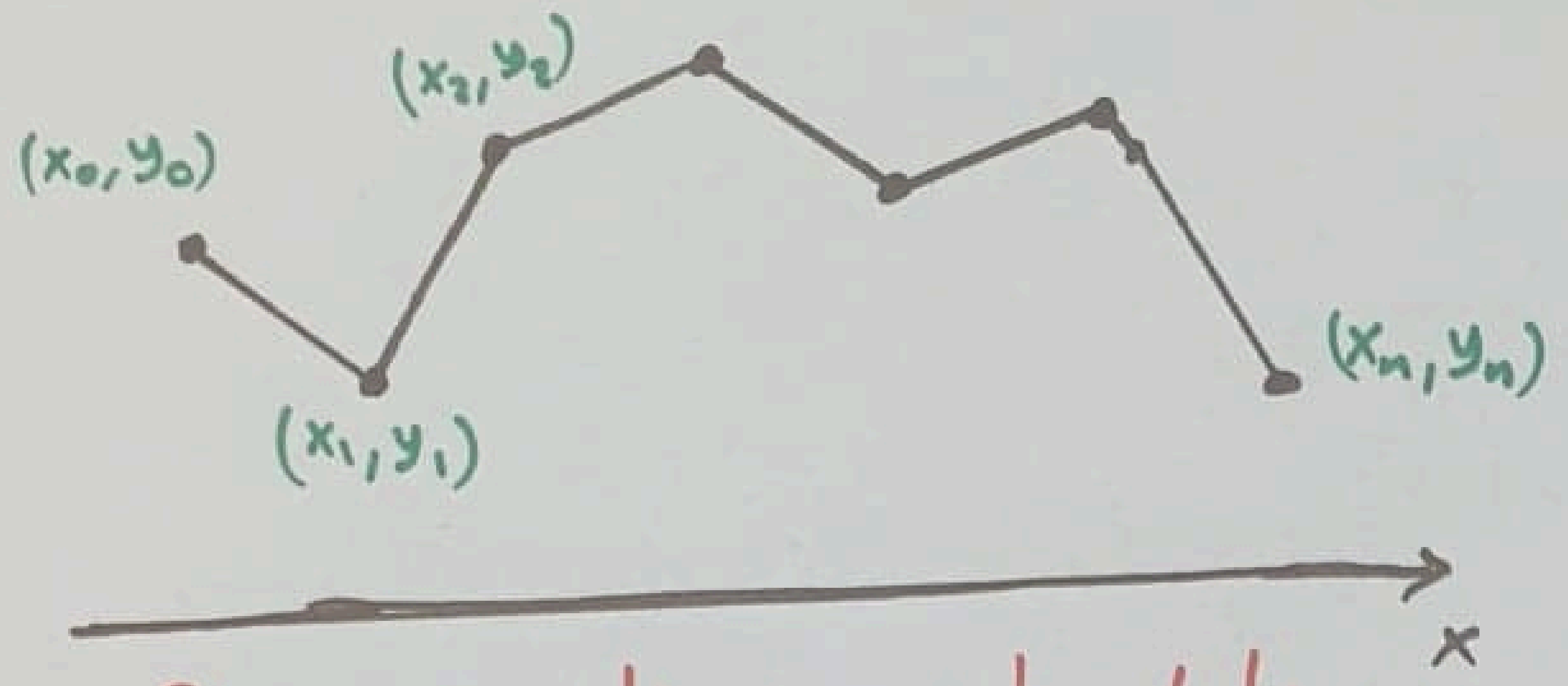
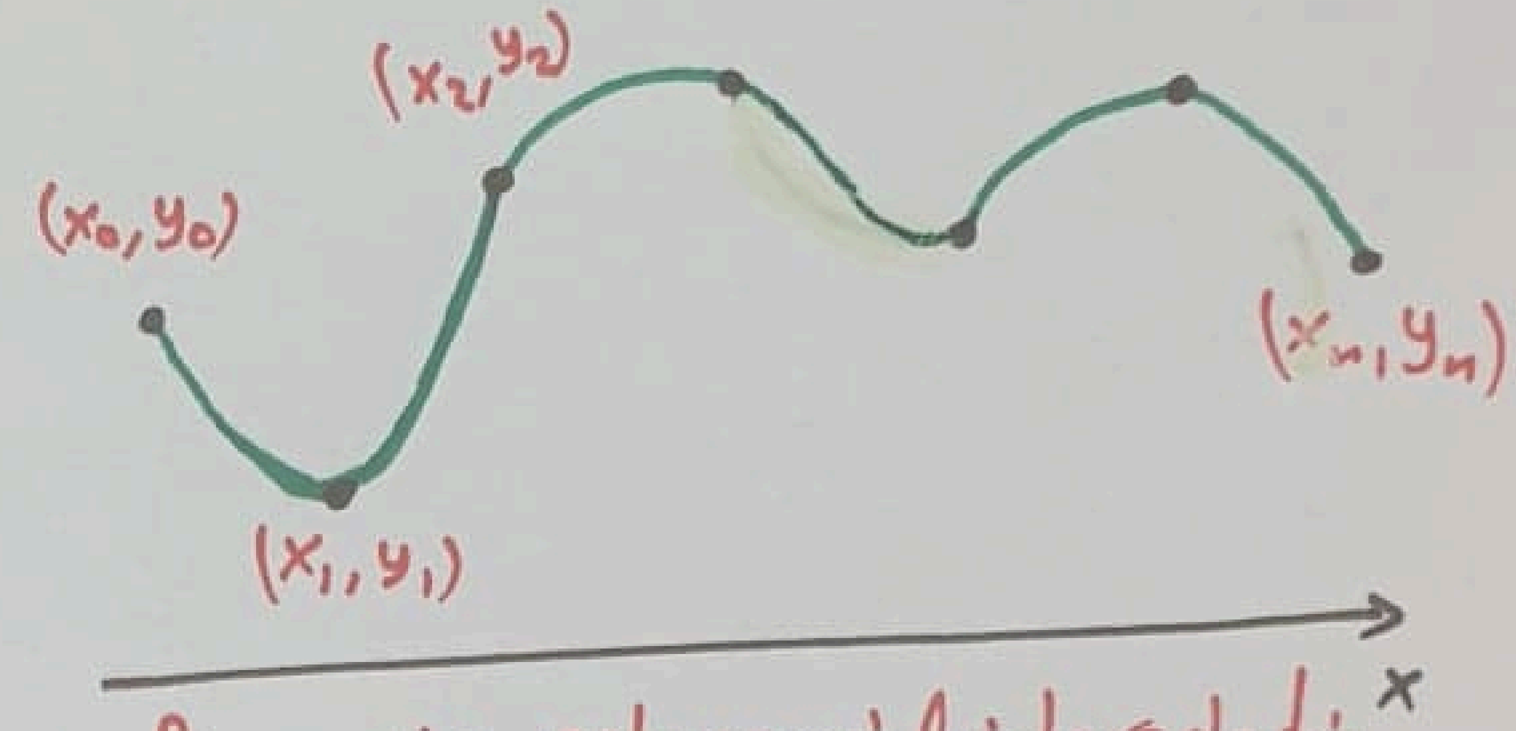
5.3 Interpolation by Spline Functions

- In this section we study a piecewise interpolation.
- Piecewise interpolation can be linear or nonlinear "polynomial" interpolation.



Piecewise linear interpolation
"linear spline"



Piecewise polynomial interpolation
"Cubic Spline"

Def (Piecewise linear spline)

- The piecewise linear curve defined on $[x_k, x_{k+1}]$ is

$$S_k(x) = y_k + d_k(x - x_k)$$

where $k = 0, 1, 2, \cdots, n-1$ and $d_k = \dfrac{y_{k+1} - y_k}{x_{k+1} - x_k}$.

- That is:

$$\begin{cases} S_0(x) = y_0 + d_0(x - x_0) & , \quad x_0 \le x \le x_1 \\ S_1(x) = y_1 + d_1(x - x_1) & , \quad x_1 \le x \le x_2 \\ \vdots & \quad \vdots \\ S_k(x) = y_k + d_k(x - x_k) & , \quad x_k \le x \le x_{k+1} \\ \vdots & \quad \vdots \\ S_{n-1}(x) = y_{n-1} + d_{n-1}(x - x_{n-1}) & , \quad x_{n-1} \le x \le x_n \end{cases}$$

Remark: The Lagrange polynomial is used to represent this piecewise linear spline: $S_k(x) = y_k \dfrac{x - x_{k+1}}{x_k - x_{k+1}} + y_{k+1} \dfrac{x - x_k}{x_{k+1} - x_k}$ for $x_k \le x \le x_{k+1}$

where $k = 0, 1, 2, \cdots, n-1$

Def (Piecewise Cubic Splines)

- Given $n+1$ points: $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
- The function $S(x)$ defined by n formula on $[a,b] = [x_0, x_n]$:

$$S(x) = \begin{cases} S_0(x) = A_0(x-x_0)^3 + B_0(x-x_0)^2 + C_0(x-x_0) + D_0, & x_0 \le x \le x_1 \\ S_1(x) = A_1(x-x_1)^3 + B_1(x-x_1)^2 + C_1(x-x_1) + D_1, & x_1 \le x \le x_2 \\ \vdots & \vdots \\ S_{n-1}(x) = A_{n-1}(x-x_{n-1})^3 + B_{n-1}(x-x_{n-1})^2 + C_{n-1}(x-x_{n-1}) + D_{n-1}, & x_{n-1} \le x \le x_n \end{cases}$$

is called cubic spline iff the following conditions hold:

$\boxed{1}$
$S_0(x_0) = y_0$
$S_1(x_1) = y_1$
$S_2(x_2) = y_2$
$\vdots$
$S_{n-1}(x_{n-1}) = y_{n-1}$
$S_{n-1}(x_n) = y_n$

$n+1$ conditions (equations)

$\boxed{2}$
$S_0(x_1) = S_1(x_1)$
$S_1(x_2) = S_2(x_2)$
$\vdots$
$S_{n-2}(x_{n-1}) = S_{n-1}(x_{n-1})$

$n-1$ conditions

$\boxed{3}$
$S_0'(x_1) = S_1'(x_1)$
$S_1'(x_2) = S_2'(x_2)$
$\vdots$
$S_{n-2}'(x_{n-1}) = S_{n-1}'(x_{n-1})$

$n-1$ conditions

$\boxed{4}$
$S_0''(x_1) = S_1''(x_1)$
$S_1''(x_2) = S_2''(x_2)$
$\vdots$
$S_{n-2}''(x_{n-1}) = S_{n-1}''(x_{n-1})$

$n-1$ conditions

Remark: we use cubic splines to estimate $f(x)$ on $[a,b] = [x_0, x_n]$

∵ Cubic splines produces $4n-2$ equations but we have $4n$ unknows so there is two degree of freedom (2 missing conditions).

∵ We use cubic splines to estimate $f(x)$ because we can make its first and second derivatives all continuous on the large interval $[x_0, x_n]$ so that $S(x) = y$ has no sharp corners.

• Depending on the remaining two conditions, there are two types of cubic spline :

1. Clamped Cubic Spline : $S'(a) = S'_0(x_0) = f'(x_0)$

$S'(b) = S'_{n-1}(x_n) = f'(x_n)$

2. Natural Cubic Spline : $S''(a) = S''_0(x_0) = 0$

$S''(b) = S''_{n-1}(x_n) = 0$

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n=1

Exp Given (x_0, y_0) and (x_1, y_1). Write the form of the cubic spline $S(x)$ that estimat $y = f(x)$.

$$S(x) = S_0(x) = A_0 (x - x_0)^3 + B_0 (x - x_0)^2 + C_0 (x - x_0) + D_0 ,$$

$$x_0 \le x \le x_1$$

since $n = 1$

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n=2

Exp Given the following data points: $\overset{x_0}{(1,2)}, \overset{x_1}{(2,3)}, \overset{x_2}{(3,5)}$.

i. Find the natural cubic spline through these data.

ii. Find the clamped cubic spline through these data given that $f'(1) = 2$ and $f'(3) = 1$

• $n = 2 \Rightarrow$

$$S(x) = \begin{cases} S_0(x) = A_0 (x-1)^3 + B_0 (x-1)^2 + C_0 (x-1) + D_0 , & 1 \le x \le 2 \\ S_1(x) = A_1 (x-2)^3 + B_1 (x-2)^2 + C_1 (x-2) + D_1 , & 2 \le x \le 3 \end{cases}$$

$$S'(x) = \begin{cases} S_0'(x) = 3A_0(x-1)^2 + 2B_0(x-1) + C_0 & , \quad 1 \le x \le 2 \\ S_1'(x) = 3A_1(x-2)^2 + 2B_1(x-2) + C_1 & , \quad 2 \le x \le 3 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 6A_0(x-1) + 2B_0 & , \quad 1 \le x \le 2 \\ S_1''(x) = 6A_1(x-2) + 2B_1 & , \quad 2 \le x \le 3 \end{cases}$$

$\boxed{1} \Rightarrow \quad S_0(x_0) = y_0 \iff S_0(1) = 2 \iff \boxed{D_0 = 2}$

$S_1(x_1) = y_1 \iff S_1(2) = 3 \iff \boxed{D_1 = 3}$

$S_1(x_2) = y_2 \iff S_1(3) = 5 \iff A_1 + B_1 + C_1 + D_1 = 5$

$\iff \boxed{A_1 + B_1 + C_1 = 2} - *^1$

$\boxed{2} \Rightarrow \quad S_0(x_1) = S_1(x_1) \iff S_0(2) = S_1(2) \iff A_0 + B_0 + C_0 + D_0 = D_1$

$\iff \boxed{A_0 + B_0 + C_0 = 1} - *^2$

$\boxed{3} \Rightarrow \quad S_0'(x_1) = S_1'(x_1) \iff S_0'(2) = S_1'(2) \iff \boxed{3A_0 + 2B_0 + C_0 = C_1} - *^3$

$\boxed{4} \Rightarrow \quad S_0''(x_1) = S_1''(x_1) \iff S_0''(2) = S_1''(2) \iff \boxed{6A_0 + 2B_0 = 2B_1} - *^4$

$\boxed{i}$ For Natural Cubic spline $\Rightarrow$

$S''(a) = S_0''(x_0) = S_0''(1) = 0 \iff 2B_0 = 0 \iff \boxed{B_0 = 0}$

$S''(b) = S_1''(x_2) = S_1''(3) = 0 \iff 6A_1 + 2B_1 = 0 \iff \boxed{3A_1 + B_1 = 0} - *^5$

$*^2 \Rightarrow C_0 = 1 - A_0$ so $*^3$ becomes $3A_0 + 1 - A_0 = C_1$

$2A_0 + 1 = C_1$

$*^4 \Rightarrow B_1 = 3A_0$ so $*^1$ becomes $A_1 + 3A_0 + 2A_0 + 1 = 2$

$$\left.\begin{aligned} A_1 + 5A_0 &= 1 \\ *^5 \text{ becomes } A_1 + A_0 &= 0 \end{aligned}\right\} \rightarrow \begin{aligned} A_0 &= \frac{1}{4} \\ A_1 &= -\frac{1}{4} \end{aligned}$$

$C_0 = 1 - A_0 = \frac{3}{4}$

$C_1 = 2A_0 + 1 = \frac{3}{2}$

$B_1 = 3A_0 = \frac{3}{4}$

Hence, the natural cubic spline is

$$s(x) = \begin{cases} s_0(x) = \frac{1}{4}(x-1)^3 + \frac{3}{4}(x-1) + 2, & 1 \le x \le 2 \\ s_1(x) = -\frac{1}{4}(x-2)^3 + \frac{3}{4}(x-2)^2 + \frac{3}{2}(x-2) + 3, & 2 \le x \le 3 \end{cases}$$

ii For clamped cubic spline $\Rightarrow$

- $s'(a) = s_0'(x_0) = s_0'(1) = f'(1) = 2 \Leftrightarrow s_0'(1) = 2$
$\Leftrightarrow \boxed{C_0 = 2}$

- $s'(b) = s_1'(x_2) = s_1'(3) = f'(3) = 1 \Leftrightarrow s_1'(3) = 1$
$\Leftrightarrow \boxed{3A_1 + 2B_1 + C_1 = 1}$ $*6$

- substitute $C_0 = 2$ in $*^1, *^2, *^3, *^4$ and add $*6 \Rightarrow$

$$\left.\begin{array}{ll} A_1 + B_1 + C_1 = 2 & \cdots *^1 \\ A_0 + B_0 + C_0 = 1 & \cdots *^2 \\ 3A_0 + 2B_0 + C_0 - C_1 = 0 & \cdots *^3 \\ 6A_0 + 2B_0 - 2B_1 = 0 & \cdots *^4 \end{array}\right\} \Rightarrow \begin{array}{l} A_1 + B_1 + C_1 = 2 \\ A_0 + B_0 = -1 \\ 3A_0 + 2B_0 - C_1 = -2 \\ 6A_0 + 2B_0 - 2B_1 = 0 \\ 3A_1 + 2B_1 + C_1 = 1 \end{array}$$

- Write this system using matrix form with order
A_0, B_0, A_1, B_1, C_1

$$\begin{array}{cccccc} A_0 & B_0 & A_1 & B_1 & C_1 & b \end{array}$$
$$\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 3 & 2 & 0 & 0 & -1 & -2 \\ 6 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right]$$

Pivot

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 3 & 2 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 6 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ \\ R_4 - 6R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & -4 & 0 & -2 & 0 & 6 \\ 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right]\begin{array}{l} \\ \\ \\ R_4 - 4R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & -2 & 4 & 2 \\ 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right]\begin{array}{l} \\ -R_2 \\ \\ R_4/-2 \\ R_5 - 3R_3 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -1 & -2 & -5 \end{array}\right]\begin{array}{l} \\ \\ \\ \\ R_5 + R_4 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & -4 & -6 \end{array}\right]$$

$-4C_1 = -6 \Rightarrow \boxed{C_1 = \frac{3}{2}}$

$B_1 - \cancel{2}\left(\frac{3}{\cancel{2}}\right) = -1$

$\boxed{B_1 = 2}$

$A_1 + \cancel{(2)} + \left(\frac{3}{2}\right) = \cancel{2} \Rightarrow \boxed{A_1 = \frac{-3}{2}}$, $B_0 + \frac{3}{2} = -1 \Rightarrow B_0 = \frac{-5}{2}$

$A_0 + \left(\frac{-5}{2}\right) = -1 \Rightarrow A_0 = \frac{3}{2}$

Hence, the clamped cubic spline is

$$s(x) = \begin{cases} S_0(x) = \frac{3}{2}(x-1)^3 - \frac{5}{2}(x-1)^2 + 2(x-1) + 2, & 1 \le x \le 2 \\ S_1(x) = -\frac{3}{2}(x-2)^3 + 2(x-2)^2 + \frac{3}{2}(x-2) + 3, & 2 \le x \le 3 \end{cases}$$

Exp • Given $\overset{x_0\ y_0}{(0,1)}, \overset{x_1\ y_1}{(1,2)}, \overset{x_2\ y_2}{(3,4)}$

• Find the Natural Cubic Spline

• $n=2$ $\Rightarrow$ The cubic spline is

$$S(x) = \begin{cases} S_0(x) = A_0 x^3 + B_0 x^2 + C_0 x + D_0, & 0 \le x \le 1 \\ S_1(x) = A_1 (x-1)^3 + B_1 (x-1)^2 + C_1 (x-1) + D_1, & 1 \le x \le 3 \end{cases}$$

$$S'(x) = \begin{cases} S_0'(x) = 3A_0 x^2 + 2B_0 x + C_0, & 0 \le x \le 1 \\ S_1'(x) = 3A_1 (x-1)^2 + 2B_1 (x-1) + C_1, & 1 \le x \le 3 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 6A_0 x + 2B_0, & 0 \le x \le 1 \\ S_1''(x) = 6A_1 (x-1) + 2B_1, & 1 \le x \le 3 \end{cases}$$

1. $S_0(x_0) = y_0 \iff S_0(0) = 1 \iff \boxed{D_0 = 1}$

   $S_1(x_1) = y_1 \iff S_1(1) = 2 \iff \boxed{D_1 = 2}$

   $S_1(x_2) = y_2 \iff S_1(3) = 4 \iff 8A_1 + 4B_1 + 2C_1 + 2 = 4$

   $\iff \boxed{4A_1 + 2B_1 + C_1 = 1} - *^1$

2. $S_0(x_1) = S_1(x_1) \iff S_0(1) = S_1(1)$

   $\iff A_0 + B_0 + C_0 + 1 = 2$

   $\iff \boxed{A_0 + B_0 + C_0 = 1} - *^2$

3. $S_0'(x_1) = S_1'(x_1) \iff S_0'(1) = S_1'(1) \iff \boxed{3A_0 + 2B_0 + C_0 = C_1} - *^3$

4. $S_0''(x_1) = S_1''(x_1) \iff S_0''(1) = S_1''(1) \iff 6A_0 + 2B_0 = 2B_1$

   $\iff \boxed{3A_0 + B_0 = B_1} - *^4$

For natural cubic spline $\Rightarrow$

$$S''(a) = S_0''(x_0) = S_0''(0) = 0 \quad \Leftrightarrow \quad \boxed{B_0 = 0}$$

$$S''(b) = S_1''(x_2) = S_1''(3) = 0 \quad \Leftrightarrow \quad 12A_1 + 2B_1 = 0$$

$$\Leftrightarrow \quad \boxed{B_1 = -6A_1} \; *^5$$

- Solving $*^1, *^2, *^3, *^4, *^5$ gives $\boxed{A_0 = A_1 = B_1 = 0}$ and $\boxed{C_0 = C_1 = 1}$
- Hence, the natural cubic spline becomes linear:

$$S(x) = \begin{cases} S_0(x) = x + 1 & , 0 \le x \le 1 \\ S_1(x) = x + 1 & , 1 \le x \le 3 \end{cases}$$

$$= x + 1 \quad \text{on} \quad 0 \le x \le 3$$

Exp Consider the following function:

$$S(x) = \begin{cases} S_0(x) = x^3 + x - 1 & , 0 \le x \le 1 \\ S_1(x) = 1 + C(x-1) + D(x-1)^2 - (x-1)^3 & , 1 \le x \le 2 \end{cases}$$

[a] Find the constats C and D that makes S(x) cubic spline.

[b] Is S(x) natural cubic spline?

$$S'(x) = \begin{cases} S_0'(x) = 3x^2 + 1 & , 0 \le x \le 1 \\ S_1'(x) = C + 2D(x-1) - 3(x-1)^2 & , 1 \le x \le 2 \end{cases}$$

$$S''(x) = \begin{cases} S_0''(x) = 6x & , 0 \le x \le 1 \\ S_1''(x) = 2D - 6(x-1) & , 1 \le x \le 2 \end{cases}$$

[a] • $S(x)$ is continuous at $x_1 = 1$ $\Leftrightarrow S_0(1) = S_1(1)$

$\Leftrightarrow 1 = 1$ does not help

• $S(x)$ is differentiable at $x_1 = 1$ $\Leftrightarrow S_0'(1) = S_1'(1)$

$\Leftrightarrow \boxed{4 = C}$

• $S(x)$ is twice diff. at $x_1 = 1$ $\Leftrightarrow S_0''(1) = S_1''(1)$

$\Leftrightarrow 6 = 2D \Leftrightarrow \boxed{D = 3}$

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[b] We check if $S_0''(x_0) = S_0''(0) \stackrel{?}{=} 0 \Rightarrow S_0''(0) = (6)(0) = 0$

and $S_1''(x_2) = S_1''(2) \stackrel{?}{=} 0 \Rightarrow S_1''(2) = 2(3) - 6(2-1) = 0$

so the cubic spline $S(x)$ is natural.