Ch 6 Numerical Differentiation

* In this chapter, we study formulas to approximate the derivatives $f'(x_0)$, $f''(x_0)$, $f'''(x_0)$, ...

* For example
  - We know $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$
  - Here h is a step size
  - This limit gives exact value for $f'(x_0)$
  - we need to replace $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ by a Difference Formula (D.F) to approximate $f'(x_0)$.

---

* We study three types of Difference Formulas:
  1. Central Difference formula (C.D.F)
  2. Backward Difference Formula (B.D.F)
  3. Forward Difference Formula (F.D.F)

---

* Notation
  - $f_k = f(x_0 + kh)$, $k = 0, \pm 1, \pm 2, \pm 3, \ldots$
  - That is,

$$
\begin{aligned}
f_0 &= f(x_0) \\
f_1 &= f(x_0 + h) \\
f_{-1} &= f(x_0 - h) \\
f_2 &= f(x_0 + 2h) \\
f_{-2} &= f(x_0 - 2h) \\
&\vdots
\end{aligned}
$$

D.F for $f'(x_0)$

1 C.D.F of order $o(h^2)$:

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h} + E_{trun}(f,h)$$

$$= \frac{f_1 - f_{-1}}{2h} + \frac{-h^2 f'''(c)}{6}$$

2 F.D.F of order $o(h^2)$:

$$f'(x_0) \approx \frac{-3f_0 + 4f_1 - f_2}{2h} + \frac{h^2 f'''(c)}{3}$$

3 B.D.F of order $o(h^2)$

$$f'(x_0) \approx \frac{3f_0 - 4f_{-1} + f_{-2}}{2h} + \frac{h^2 f'''(c)}{3}$$

4 C.D.F of order $o(h^4)$

$$f'(x_0) \approx \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$$

Remark
- $E_{trun}(f,h)$ is called the truncation error.
- In case of 1, 2, 3 $\Rightarrow$ f is assumed to be $C^3[a,b]$
- In case of 4 $\Rightarrow$ f is assumed to be $C^5[a,b]$
- And so on ...

D.F for $f''(x_0)$

1 C.D.F of order $o(h^2)$:

$$f''(x_0) \approx \frac{f_1 - 2f_0 + f_{-1}}{h^2} + \frac{-h^2 f^{(4)}(c)}{12}$$

2 F.D.F of order $o(h^2)$:

$$f''(x_0) \approx \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} + \frac{11 h^2 f^{(4)}(c)}{12}$$

3 B.D.F of order $o(h^2)$

$$f''(x_0) \approx \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2} + \frac{11 h^2 f^{(4)}(c)}{12}$$

4 C.D.F of order $o(h^4)$:

$$f''(x_0) = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} + \frac{h^4 f^{(6)}(c)}{90}$$

Remark To derive any of these D.F's for $f'(x_0)$ and $f''(x_0)$, we use:

① Taylor's Expansion or

② Lagrange Interpolation or

③ Newton Interpolation.

127.1

D.F for $f'(x_0)$

[1] C.D.F of order $O(h^2)$: $f'(x_0) = \frac{f_1 - f_{-1}}{2h} - \frac{h^2 f'''(c)}{6}$

[2] F.D.F of order $O(h^2)$: $f'(x_0) = \frac{-3f_0 + 4f_1 - f_2}{2h} + \frac{h^2 f'''(c)}{3}$

[3] B.D.F of order $O(h^2)$: $f'(x_0) = \frac{3f_0 - 4f_{-1} + f_{-2}}{2h} + \frac{h^2 f'''(c)}{3}$

[4] C.D.F of order $O(h^4)$: $f'(x_0) = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$

D.F for $f''(x_0)$

[1] C.D.F of order $O(h^2)$: $f''(x_0) = \frac{f_1 - 2f_0 + f_{-1}}{h^2} - \frac{h^2 f^{(4)}(c)}{12}$

[2] F.D.F of order $O(h^2)$: $f''(x_0) = \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} + \frac{11h^2 f^{(4)}(c)}{12}$

[3] B.D.F of order $O(h^2)$: $f''(x_0) = \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2} + \frac{11h^2 f^{(4)}(c)}{12}$

[4] C.D.F of order $O(h^4)$: $f''(x_0) = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} + \frac{h^4 f^{(6)}(c)}{90}$

Exp Consider the following points:

$(2,-1), (2.1, 2), (2.2, -1.5), (2.3, 0), (2.4, 2)$

1. Estimate $f'(2.1)$ using C.D.F of order $O(h^2)$
2. Estimate $f'(2.2)$ using F.D.F of order $O(h^2)$

Clearly $h = 0.1$

1. $$f'(2.1) \approx \frac{f_1 - f_{-1}}{2h} = \frac{f(2.1+0.1) - f(2.1-0.1)}{2(0.1)}$$
$$= \frac{f(2.2) - f(2)}{0.2} = \frac{-1.5 - -1}{0.2} = \frac{-0.5}{0.2} = -2.5$$

2. $$f'(2.2) \approx \frac{-3f_0 + 4f_1 - f_2}{2h} = \frac{-3f(2.2) + 4f(2.3) - f(2.4)}{2(0.1)}$$
$$= \frac{-3(-1.5) + 4(0) - 2}{0.2} = \frac{2.5}{0.2} = 12.5$$

Exp Let $f(x) = e^x$. Estimate $f'(1)$ using B.D.F of order $O(h^2)$ with (i) $h = 0.01$ (ii) $h = 0.001$

$$f'(1) \approx \frac{3f_0 - 4f_{-1} + f_{-2}}{2h} = \frac{3f(1) - 4f(1-h) + f(1-2h)}{2h}$$

(i) $$f'(1) = \frac{3f(1) - 4f(0.99) + f(0.98)}{0.02} = \frac{3e - 4e^{0.99} + e^{0.98}}{0.02}$$
$$= 2.7181918955$$

(ii) $$f'(1) = \frac{3f(1) - 4f(0.999) + f(0.998)}{0.02} = \frac{3e - 4e^{0.999} + e^{0.998}}{0.002}$$
$$= 2.718280923$$

Exp Let $f(x) = \sin x$

1. Estimate $f'(0)$ using C.D.F of order $O(h^4)$ with step size $h = 0.01$
2. Estimate $f'(3)$ using B.D.F of order $O((0.1)^2)$.

1. $$f'(0) \approx \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h}$$

$$= \frac{-f(0.02) + 8f(0.01) - 8f(-0.01) + f(-0.02)}{12(0.01)}$$

$$= \frac{-2\sin(0.02) + 16\sin 0.01}{0.12} \approx \frac{0.120061199}{0.12} = 1.00051$$

2. $h = 0.1$, $x_0 = 3$ $\Rightarrow$ $$f'(3) \approx \frac{3f_0 - 4f_{-1} + f_{-2}}{2h}$$

$$= \frac{3f(3) - 4f(2.9) + f(2.8)}{2(0.1)}$$

$$= \frac{3\sin 3 - 4\sin(2.9) + \sin(2.8)}{0.2}$$

$$= -0.9939715075$$

Note that the true values are:

$f'(x) = \cos x$

$f'(0) = \cos 0 = 1$

$f'(3) = \cos 3 = -0.9902072488$

Exp Consider the following table:

t	1	1.3	1.6	1.9
D	10	30	60	100

where t : time
D : distance

1. Estimate the velocity at $t = 1.6$ using C.D.F of $O(h^2)$
2. Estimate the acceleration at $t = 1.3$ using C.D.F of $O(h^2)$
3. Estimate the velocity at $t = 1.6$ using F.D.F of $O(h^2)$.

$h = 0.3 \Rightarrow$

1. $$v(t_0) = D'(t_0) \approx \frac{D_1 - D_{-1}}{2h} = \frac{D(t_0+h) - D(t_0-h)}{2h}$$

$t_0 = 1.6$

$$D'(1.6) \approx \frac{D(1.6+0.3) - D(1.6-0.3)}{2(0.3)} = \frac{D(1.9) - D(1.3)}{0.6}$$

$$= \frac{100-30}{0.6} = \frac{70}{0.6} = 116.67$$

---

2. $$a(t_0) = D''(t_0) \approx \frac{D_1 - 2D_0 + D_{-1}}{h^2} = \frac{D(1.6) - 2D(1.3) + D(1)}{(0.3)^2}$$

$t_0 = 1.3$

$$= \frac{60 - 2(30) + 10}{0.09} = 111.11$$

---

3. $$v(t_0) = D'(t_0) \approx \frac{-3D_0 + 4D_1 - D_2}{2h}$$ ✓ ✓ ✗ not possible

$t_0 = 1.6$