Error Analysis and Optimal step Size

- In any D.F. $\Rightarrow$

  Total Error = Round off Error + Truncation Error

$$\boxed{E_{Total}(f,h) = E_{round}(f,h) + E_{trun}(f,h)}$$

- The Round off Error $E_{round}(f,h)$ in any D.F :

$$f_k = y_k + e_k \quad , \; k = 0, \pm 1, \pm 2, \dots$$

$$\Rightarrow f_0 = y_0 + e_0 \; , \; f_1 = y_1 + e_1 \; , \; f_2 = y_2 + e_2$$
$$f_{-1} = y_{-1} + e_{-1} \; , \; f_{-2} = y_{-2} + e_{-2} \quad \dots$$

- **Remark**: The magnitude of the round off error is

$$|e_k| \leq \epsilon = 5 \times 10^{-10}$$

- The truncation error $E_{trun}(f,h)$ depends on the form of the D.F.
- To find the optimal step size h for a given D.F, we differentiate $E_{Total}(f,h)$ w.r.t h and find the critical value.

*Exp. Let $f(x) = \sin x$

- Estimate $f'(1)$ using the C.D.F of order $O(h^2)$ with $h = 0.01$ and $h = 0.001$ and $h = 0.0001$ and compare with the true value.

- True Value : $f'(x) = \cos x \Rightarrow f'(1) = \boxed{0.5403023059}$
- $f'(x_0) \simeq \dfrac{f(x_0+h) - f(x_0-h)}{2h}$
- $h = 0.01 \Rightarrow f'(1) = \dfrac{f(1.01) - f(0.99)}{2(0.01)} = 0.5402933009$
- $h = 0.001 \Rightarrow f'(1) = \dfrac{f(1.001) - f(0.999)}{2(0.001)} = 0.5403022158$
- $\boxed{h = 0.0001} \Rightarrow f'(1) = \dfrac{f(1.0001) - f(0.9999)}{2(0.0001)} = \boxed{0.540302305}$

  optimal since it gives zero error

Exp Find the optimal step size h for the C.D.F of order $O(h^2)$ using to estimate $f'(x_0)$.

- $$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + E_{trun}(f,h)$$

$$= \frac{f_1 - f_{-1}}{2h} - \frac{h^2 f'''(c)}{6}$$

$$= \frac{y_1 + e_1 - (y_{-1} + e_{-1})}{2h} - \frac{h^2 f'''(c)}{6}$$

$$= \frac{y_1 - y_{-1}}{2h} + \underbrace{\frac{e_1 - e_{-1}}{2h}}_{\text{Roundoff Error}} + \underbrace{\frac{-h^2 f'''(c)}{6}}_{\text{Truncation Error}}$$

- Hence, the total error is

$$E_{total}(f,h) = E_{round\,off}(f,h) + E_{trun}(f,h)$$

$$= \frac{e_1 - e_{-1}}{2h} + \frac{-h^2 f'''(c)}{6}$$

- Now $|E_{total}| \leq \left|\frac{e_1 - e_{-1}}{2h}\right| + \left|\frac{h^2 f'''(c)}{6}\right|$

$$\leq \left|\frac{e_1}{2h}\right| + \left|\frac{e_{-1}}{2h}\right| + \frac{h^2 M_3}{6}$$

$$\leq \frac{\epsilon}{2h} + \frac{\epsilon}{2h} + \frac{h^2 M_3}{6} \quad \text{by Remark page 136}$$

$$= \frac{\epsilon}{h} + \frac{h^2 M_3}{6}$$

$$= \phi(h)$$

- Now set $\phi'(h) = 0$ and find the critical value h^*

$$\frac{-\epsilon}{h^2} + \frac{h M_3}{3} = 0 \iff h^3 M_3 = 3\epsilon$$

$$\iff h^* = \left(\frac{3\epsilon}{M_3}\right)^{\frac{1}{3}}$$

- Note that in *Exp $\Rightarrow M_3 = \max|f'''(c)| = 1$ since $f(x) = \sin x$

$$\Rightarrow h^* = (3\epsilon)^{\frac{1}{3}} = (3 \times 5 \times 10^{-10})^{\frac{1}{3}} \simeq 0.001145 > 0.001 \checkmark$$

$$\Rightarrow h^* \approx 0.0001 \checkmark \text{ better}$$

- Remember that for C.D.F of order $O(h^2)$ to estimate $f'(x_0) \Rightarrow$ we have $f'(x_0) = \frac{f_1 - f_{-1}}{2h} - \frac{h^2 f'''(c)}{6}$ with

$$\phi(h) = \frac{\epsilon}{h} + \frac{h^2 M_3}{6} \quad \text{and} \quad h^* = \left(\frac{3\epsilon}{M_3}\right)^{\frac{1}{3}}$$

Exp Find the optimal step size h for the C.D.F of order $O(h^2)$ in estimating $f''(x_0)$.

- $$f''(x_0) = \frac{f_1 - 2f_0 + f_{-1}}{h^2} + E_{trun}(f,h)$$

$$= \frac{y_1 + e_1 - 2(y_0 + e_0) + y_{-1} + e_{-1}}{h^2} + \frac{-h^2 f^{(4)}(c)}{12}$$

$$= \frac{y_1 - 2y_0 + y_{-1}}{h^2} + \underbrace{\frac{e_1 - 2e_0 + e_{-1}}{h^2}}_{\text{Round off Error}} + \underbrace{\frac{-h^2 f^{(4)}(c)}{12}}_{\text{Truncation Error}}$$

- Hence, $E_{total}(f,h) = E_{roundoff}(f,h) + E_{trun}(f,h)$

$$= \frac{e_1 - 2e_0 + e_{-1}}{h^2} + \frac{-h^2 f^{(4)}(c)}{12}$$

- Now $|E_{total}| \le \left|\frac{e_1 - 2e_0 + e_{-1}}{h^2}\right| + \left|\frac{h^2 f^{(4)}(c)}{12}\right|$

$$\le \frac{\epsilon + 2\epsilon + \epsilon}{h^2} + \frac{h^2 M_4}{12}$$

$$= \frac{4\epsilon}{h^2} + \frac{h^2 M_4}{12}$$

$$= \phi(h)$$

- $\phi'(h) = 0 \iff \frac{-8\epsilon}{h^3} + \frac{h M_4}{6} = 0 \iff$

$$h^* = \left(\frac{48\epsilon}{M_4}\right)^{\frac{1}{4}}$$

Exp • Let $f(x) = \ln x$ where $0.1 \leq x \leq 0.5$

• Find the best step size h for the C.D.F of order $O(h^2)$ in estimating $f''(x_0)$

• Using the previous Exp $\Rightarrow$ $h^* = \left(\frac{48\epsilon}{M_4}\right)^{\frac{1}{4}}$

• To find $M_4 \Rightarrow f'(x) = \frac{1}{x} \Rightarrow f''(x) = \frac{-1}{x^2}$

$$\Rightarrow f'''(x) = \frac{2}{x^3} \Rightarrow f^{(4)}(x) = \frac{-6}{x^4}$$

$$M_4 = \max_{0.1 \leq x \leq 0.5} \left| f^{(4)}(x) \right| = \max_{0.1 \leq x \leq 0.5} \frac{6}{x^4} = \frac{6}{(0.1)^4} = 60000$$

• $h^* = \left(\frac{48 \times 5 \times 10^{-10}}{60000} \right)^{\frac{1}{4}} = \left(4 \times 10^{-13}\right)^{\frac{1}{4}} = 0.0007952707$

Exp Find the optimal h for $f(x) = e^{-x}$, $1 \leq x \leq 2$ if the C.D.F of order $O(h^4)$ is used to estimate $f'(x_0)$.

• $f'(x_0) = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$

• Similarly to what have been done befor, we could arrive:

$$\phi(h) = \frac{18\epsilon}{12h} + \frac{h^4 M_5}{30} = \frac{3\epsilon}{2h} + \frac{h^4 M_5}{30}$$

• $\phi'(h) = \frac{-3\epsilon}{2h^2} + \frac{2h^3 M_5}{15} = 0 \Leftrightarrow h^* = \left(\frac{45\epsilon}{4M_5}\right)^{\frac{1}{5}}$

• To find $M_5 \Rightarrow f' = -e^{-x} = f''' = f^{(5)} \Rightarrow M_5 = \max_{1 \leq x \leq 2} \left| f^{(5)}(x) \right| = \left| \frac{1}{e^x} \right|_{x=1}$

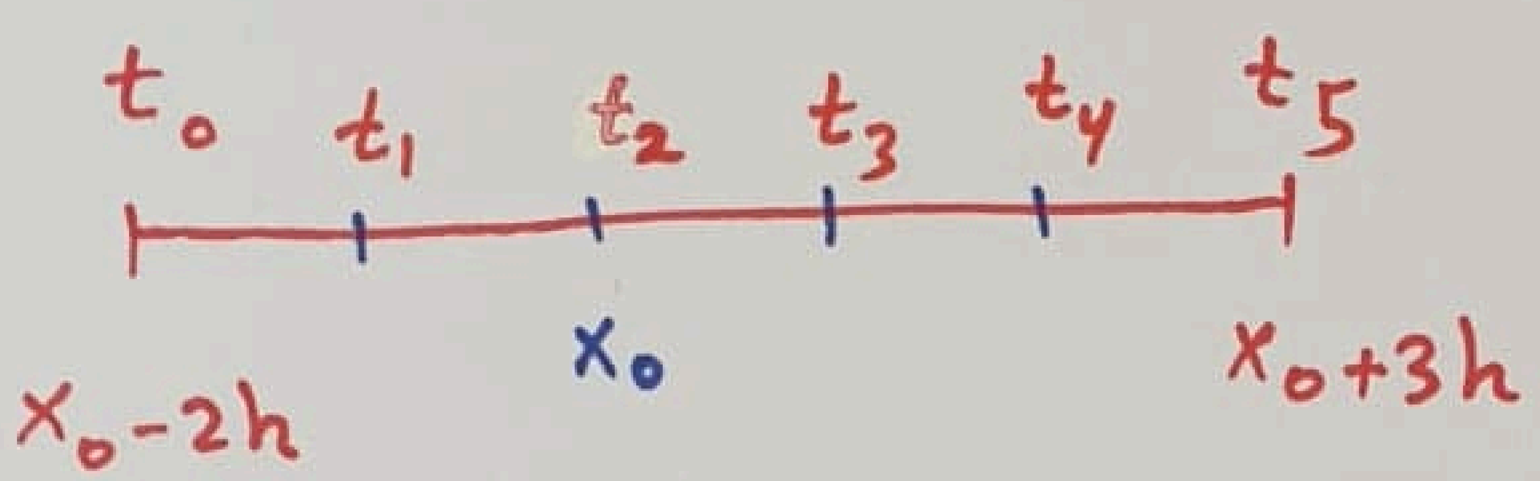
$$= \frac{1}{e} = 0.3678794412$$

• Hence, $h^* = \left(\frac{45 \times 5 \times 10^{-10}}{4(0.3678794412)} \right)^{\frac{1}{5}}$

$=$ 2.7345344669

Exp Use the points: $x_0 - 2h$, $x_0 + 3h$ to estimate $f'(x_0)$ with its truncation error using Newton's Interpolation.

- $n = 1 \Rightarrow$ Newton's Poly. is $P_1(t) = a_0 + a_1(t - t_0)$



$$P_1'(t) = a_1 = f[t_0, t_5] = \frac{f(t_5) - f(t_0)}{t_5 - t_0} = \frac{f_3 - f_{-2}}{5h} = P_1'(x_0)$$

- $f(t) = P_1(t) + E_1(t)$

$$f'(t) = P_1'(t) + E_1'(t)$$

$$f'(x_0) = P_1'(x_0) + E_1'(x_0)$$

- Note that $E_1(t) = \dfrac{f''(c)(t-t_0)(t-t_5)}{2}$

$$E_1'(t) = \frac{f''(c)}{2}\left[(t-t_0) + (t-t_5)\right]$$

$$E_1'(t_2) = E_1'(x_0) = \frac{f''(c)}{2}\left[2h + (-3h)\right]$$

$$= -\frac{h f''(c)}{2}$$

- Hence, $f'(x_0) = P_1'(x_0) + E_1'(x_0)$

$$= \frac{f_3 - f_{-2}}{5h} - \frac{h f''(c)}{2}$$

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or

t_0	t_1	t_2	t_3	t_4	t_5
x_0-2h	x_0-h	x_0	x_0+h	x_0+2h	x_0+3h

NP $\Rightarrow$ $P_1(t) = a_0 + a_1(t - t_0)$ since $n=1$

$$P_1'(t) = a_1 = f[t_0, t_5] = \frac{f(t_5) - f(t_0)}{t_5 - t_0}$$

$$= \frac{f_3 - f_{-2}}{5h}$$

$$f(t) = P_1(t) + E_1(t)$$

$$f'(t) = P_1'(t) + E_1'(t)$$

$$f'(t_2) = P_1'(t_2) + E_1'(t_2)$$

$$f'(x_0) = P_1'(x_0) + E_1'(x_0)$$

$$E_1(t) = \frac{f''(c)(t-t_0)(t-t_5)}{2}$$

$$E_1'(t) = \frac{f''(c)}{2}\left[(t-t_0) + (t-t_5)\right]$$

$$E_1'(x_0) = E_1'(t_2) = \frac{f''(c)}{2}\left[t_2 - t_0 + t_2 - t_5\right] = \frac{f''(c)}{2}\left[2h + (-3h)\right]$$

$$= \frac{-h f''(c)}{2}$$

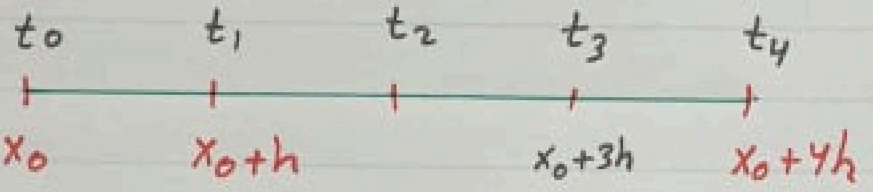
$$f'(x_0) = P_1'(x_0) + E_1'(x_0)$$

$$= \frac{f_3 - f_{-2}}{5h} - \frac{h f''(c)}{2}$$

Exp Given the nodes x_0, x_0+h, x_0+4h.
Use Newton Polynomial to find the Difference formula that estimates $f'(x_0+3h)$

• Three points $\Rightarrow n=2 \Rightarrow P_2(t) = a_0 + a_1(t-t_0) + a_2(t-t_0)(t-t_1)$

$$P_2'(t) = a_1 + a_2\left[(t-t_0)+(t-t_1)\right]$$



$$a_1 = f[t_0,t_1] = \frac{f(t_1)-f(t_0)}{t_1-t_0} = \frac{f_1-f_0}{h}$$

$$a_2 = f[t_0,t_1,t_4] = \frac{f[t_1,t_4]-f[t_0,t_1]}{t_4-t_0} = \frac{\frac{f_4-f_1}{t_4-t_1} - \frac{f_1-f_0}{t_1-t_0}}{t_4-t_0}$$

$$= \frac{\frac{f_4-f_1}{3h} - \frac{f_1-f_0}{h}}{4h} = \frac{f_4-4f_1+3f_0}{12h^2}$$

$f(t) = P_2(t) + E_2(t)$
$f'(t) \approx P_2'(t)$

$$f'(t_3) \approx P_2'(t_3) = a_1 + a_2\left((t_3-t_0)+(t_3-t_1)\right) = a_1 + a_2(3h+2h)$$

$$f'(x_0+3h) = \frac{f_1-f_0}{h} + 5\cancel{h}\left(\frac{f_4-4f_1+3f_0}{12h^{\cancel{2}}}\right)$$

$$= \frac{12f_1-12f_0}{12h} + \frac{5f_4-20f_1+15f_0}{12h}$$

$$= \frac{5f_4-8f_1+3f_0}{12h}$$