Ch 7 Numerical Integration

* How can we estimate $\int_a^b f(x)\,dx$?

We use quadrature formula $Q[f]$.

Def (Quadrature Formula)

- Suppose that $a = x_0 < x_1 < \cdots < x_m = b$.
- A formula of the form $Q[f] = \sum_{k=0}^{m} w_k f(x_k)$

$$= w_0 f(x_0) + w_1 f(x_1) + \cdots + w_m f(x_m)$$

with the property that

$$\int_a^b f(x)\,dx = Q[f] + E[f]$$

is called a numerical integration (or quadrature formula).

- $E[f]$ is called the truncation error for integration.
- $x_0, x_1, \ldots, x_m$ are called the quadrature nodes.
- $w_0, w_1, \ldots, w_m$ are called the weights.

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* We will study two types of $Q[f]$:

1. Closed Newton-Cotes Quadrature Formula:
   - (a) Trapezoidal Rule
   - (b) Simpson's Rule
   - (c) Simpson's $\frac{3}{8}$ Rule
2. Gauss-Legendre Formula:
   - (a) $G_1(f)$
   - (b) $G_2(f)$
   - (c) $G_3(f)$

Closed Newton - Coles Quadrature Formula

Th • Assume that $x_k = x_0 + kh$ are equally spaced nodes with $f_k = f(x_k)$

• Then **a** Trapezoidal Rule is

h: $a = x_0$, $b = x_1$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx \approx \underbrace{\frac{h}{2}(f_0 + f_1)}_{Q[f]} \text{ with error } \underbrace{\frac{-h^3 f''(c)}{12}}_{E[f]}$$

b Simpson's Rule is

h, h: $a = x_0$, x_1, $b = x_2$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx \approx \underbrace{\frac{h}{3}(f_0 + 4f_1 + f_2)}_{Q[f]} \text{ with error } \underbrace{\frac{-h^5 f^{(4)}(c)}{90}}_{E[f]}$$

c Simpson's 3/8 Rule is

h, h, h: $x_0 = a$, x_1, x_2, $b = x_3$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx \approx \underbrace{\frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)}_{Q[f]} \text{ with error } \underbrace{\frac{-3h^5 f^{(4)}(c)}{80}}_{E[f]}$$

Exp Estimate $\int_0^1 \left(1 + e^{-x}\sin(4x)\right)dx$ using

1 Trapezoidal Rule

$h = 1$: $x_0 = 0$, $1 = x_1$

$$\int_0^1 \left(1 + e^{-x}\sin(4x)\right)dx \approx \frac{h}{2}(f_0 + f_1)$$

$$= \frac{1}{2}(1 + 0.72159)$$

$$= 0.86079$$

$f_0 = f(0) = 1$

$f_1 = f(1) = 0.72159$

2 Simpson's Rule

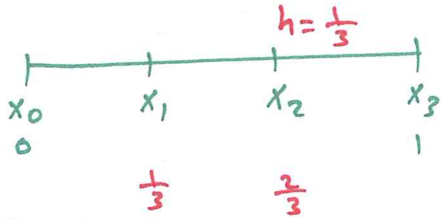
$h = 0.5$, h: $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

$$\int_0^1 \left(1 + e^{-x}\sin(4x)\right)dx \approx \frac{0.5}{3}\left(f(0) + 4f(0.5) + f(1)\right)$$

$$= \frac{1}{6}\left(1 + 4(1.55152) + 0.72159\right)$$

$$= 1.32128$$

3 Simpson's $\frac{3}{8}$ Rule



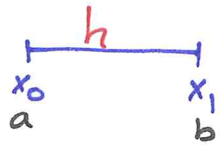
$$\int_0^1 (1+e^{-x}\sin(4x))\,dx \approx \frac{3h}{8}\left(f_0+3f_1+3f_2+f_3\right)$$

$$= \frac{3(\frac{1}{3})}{8}\left[1+3(1.69642)+3(1.23447)+0.72159\right]$$

$$= 1.31440$$

Exp Derive the Trapezoidal Rule $\int_a^b f(x) \approx \frac{h}{2}(f_0+f_1)$

- Since $a = x_0$ and $b = x_1$ $\Rightarrow$ $n = 1$



- Use Newton Interpolation $\Rightarrow$ $f(x) \approx P_1(x)$

$$P_1(x) = a_0 + a_1(x-x_0)$$

- Hence, $\displaystyle\int_a^b f(x)\,dx \approx \int_a^b P_1(x)\,dx = \int_{x_0}^{x_1}\left(a_0+a_1(x-x_0)\right)dx$

$$= a_0(x_1-x_0) + a_1 \left.\frac{(x-x_0)^2}{2}\right|_{x_0}^{x_1}$$

$$= a_0 h + \frac{a_1 h^2}{2}$$

or change of variables

$x - x_0 = ht$

$dx = h\,dt$

$x = x_0 \Rightarrow t = 0$

$x = x_1 \Rightarrow t = 1$

$$\int_{x_0}^{x_1}\left(a_0+a_1(x-x_0)\right)dx = \int_0^1 (a_0+a_1 h t)\,h\,dt$$

$$= h a_0 + \frac{a_1 h^2}{2}$$

- But $a_0 = f[x_0] = f(x_0) = f_0$

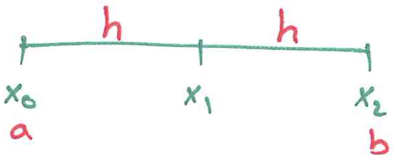
$$a_1 = f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f_1-f_0}{h}$$

- Thus, $\displaystyle\int_a^b f(x)\,dx \approx f_0 h + \frac{f_1-f_0}{2\cancel{h}}\, h^{\cancel{2}}$

$$= \frac{h}{2}\left(f_0+f_1\right)$$

EXP Derive the Simpson's Rule $\int_a^b f(x)\,dx \simeq \frac{h}{3}\left(f_0 + 4f_1 + f_2\right)$ 145

- Here $n = 2$ since we have 3 points



- Use Lagrange Poly. $\Rightarrow f(x) \approx P_2(x)$

where $P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

- Hence, $\int_a^b f(x)\,dx = \int_{x_0}^{x_2} P_2(x)\,dx$

$$= \int_{x_0}^{x_2} \frac{f_0}{2h^2}(x-x_1)(x-x_2)\,dx + \int_{x_0}^{x_2} \frac{-f_1}{h^2}(x-x_0)(x-x_2)\,dx$$

$$+ \int_{x_0}^{x_2} \frac{f_2}{2h^2}(x-x_0)(x-x_1)\,dx$$

- Use the following change of variables:

$$\left.\begin{aligned} x - x_0 &= ht \\ dx &= h\,dt \end{aligned}\right\} \Rightarrow \begin{aligned} \text{when } x = x_0 &\Rightarrow t = 0 \\ x = x_2 &\Rightarrow t = 2 \end{aligned}$$

- Note that $x - x_1 = x - (x_0 + h) = (x - x_0) - h = ht - h = h(t-1)$

$x - x_2 = x - (x_0 + 2h) = (x - x_0) - 2h = ht - 2h = h(t-2)$

- Hence, $\int_{x_0}^{x_2} P_2(x) = \int_0^2 \frac{f_0}{2\cancel{h^2}} \cancel{h}(t-1)(\cancel{h})(t-2)\,h\,dt - \int_0^2 \frac{f_1}{\cancel{h^2}}(\cancel{h}t)(\cancel{h})(t-2)\,h\,dt$

$$+ \int_0^2 \frac{f_2}{2\cancel{h^2}}(\cancel{h}t)(\cancel{h})(t-1)\,h\,dt$$

$$= \frac{hf_0}{2}\int_0^2 (t^2 - 3t + 2)\,dt - hf_1 \int_0^2 (t^2 - 2t)\,dt + \frac{hf_2}{2}\int_0^2 (t^2 - t)\,dt$$

$$= \frac{hf_0}{2}\left(\frac{t^3}{3} - \frac{3}{2}t^2 + 2t\right)\Big|_0^2 - hf_1\left(\frac{t^3}{3} - t^2\right)\Big|_0^2 + \frac{hf_2}{2}\left(\frac{t^3}{3} - \frac{t^2}{2}\right)\Big|_0^2$$

$$= \frac{hf_0}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right) - hf_1\left(\frac{-4}{3}\right) + \frac{hf_2}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right)$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right]$$