The Degree of Precision (DP)

- Recall that the quadrature formula:

$$\int_a^b f(x)\,dx = Q[f] + E[f]$$

- To derive the truncation error $E[f]$ for any quadrature formula $Q[f]$, we first study the Degree of Precision (DP) for this quadrature formula $Q[f]$.

Def. The DP of a quadrature formula $Q[f]$ is a positive integer n s.t $Q[f]$ is exact "$E[f]=0$" for $f_k = x^k$ where $k = 0, 1, 2, \dots, n$

- That is: $E[f_0] = E[x^0] = E[1] = \int_a^b dx - Q[1] = 0$

$$E[f_1] = E[x] = \int_a^b x\,dx - Q[x] = 0$$

$$E[f_2] = E[x^2] = \int_a^b x^2\,dx - Q[x^2] = 0$$

$$\vdots$$

$$E[f_n] = E[x^n] = \int_a^b x^n\,dx - Q[x^n] = 0$$

But $E[f_{n+1}] = E[x^{n+1}] = \int_a^b x^{n+1}\,dx - Q[x^{n+1}] \neq 0$

- In this case we have $n =$ DP of $Q[f]$

- We use n to find the truncation error $E[f]$ which has the general form:

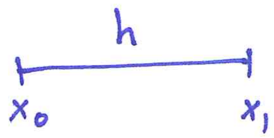
$* \dots$ $E[f] = K f^{(n+1)}(c)$ where $c \in [a,b]$ and

K is a constant that depends on the DP $= n$ and h.

Exp Determine the DP of the Trapezoidal Rule and use it to find the truncation error.

- Recall that the Trapezoidal Rule is :

$$\int_a^b f(x)\,dx = Q[f] + E[f]$$
$$= \frac{h}{2}\left[f_0 + f_1\right] + \frac{-h^3 f''(c)}{12}$$



- It will be enough to apply Trapezoidal Rule over the interval $[0,1] \Rightarrow$

$$\int_0^1 f(x)\,dx = \frac{1}{2}\left[f(0) + f(1)\right] + \frac{-h^3 f''(c)}{12}$$

- $\int_0^1 dx = 1 = \frac{1}{2}[1+1]$ with $E[1] = 0$ since $f = 1$

$\int_0^1 x\,dx = \frac{1}{2} = \frac{1}{2}[0+1]$ with $E[x] = 0$ since $f = x$

$\int_0^1 x^2\,dx = \frac{1}{3} \neq \frac{1}{2}[0+1]$ with $E[x^2] = \frac{1}{3} - \frac{1}{2} = \frac{-1}{6} \neq 0$ since $f = x^2$

- Hence the DP $= n = 1$ for the Trapezoidal Rule.
- And so, by $*$ we have $E = K f^{(n+1)}(c) = K f''(c)$
- Now to find K, we consider $f(x) = (x - x_0)^{n+1} = (x - x_0)^2$

$\Rightarrow f'(x) = 2(x - x_0) \Rightarrow f''(x) = 2 \Rightarrow \boxed{E = 2K}$ ①

- But $E = \text{True} - \text{Estimate} = \int_{x_0}^{x_1} (x - x_0)^2\,dx - \frac{h}{2}\left(f(x_0) + f(x_1)\right)$

$$= \frac{(x - x_0)^3}{3}\Bigg|_{x_0}^{x_1} - \frac{h}{2}\left(0 + (x_1 - x_0)^2\right)$$

$$= \frac{h^3}{3} - \frac{h^3}{2}$$

$$\boxed{E = \frac{-h^3}{6}}$$ ②

From ① and ② we have $2K = \frac{-h^3}{6}$

$\Leftrightarrow \boxed{K = \frac{-h^3}{12}}$ and hence, $E = K f''(c)$

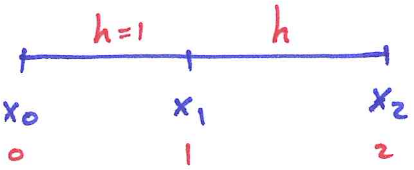
$$= \frac{-h^3}{12} f''(c)$$

Exp Determine the DP of the Simpson's Rule and use it to find the truncation error.

- Recall that the Simpson's Rule is:

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] + \frac{-h^5 f^{(4)}(c)}{90}$$

- It will be enough to apply Simpson's Rule over the interval $[0,2]$ $\Rightarrow$



$$\int_0^2 f(x)\,dx = \frac{1}{3}\left[f(0) + 4f(1) + f(2)\right] - \frac{h^5 f^{(4)}(c)}{90}$$

- $\int_0^2 dx = 2 = \frac{1}{3}[1+4+1]$ with $E[1] = 0$ since $f = 1$

$\int_0^2 x\,dx = 2 = \frac{1}{3}[0+4+2]$ with $E[x] = 0$ since $f = x$

$\int_0^2 x^2\,dx = \frac{8}{3} = \frac{1}{3}[0+4+4]$ with $E[x^2] = 0$ since $f = x^2$

$\int_0^2 x^3\,dx = 4 = \frac{1}{3}[0+4+8]$ with $E[x^3] = 0$ since $f = x^3$

$\int_0^2 x^4\,dx = \frac{32}{5} \neq \frac{1}{3}[0+4+16] = \frac{20}{3}$ with $E[x^4] = \frac{32}{5} - \frac{20}{3} \neq 0$ since $f = x^4$

- Hence, the $DP = n = 3$ for the Simpson's Rule.
- And so, by $*$ the truncation error is

$$E = K f^{(n+1)}(c) = K f^{(4)}(c)$$

- Now to find K, we consider $f(x) = (x-x_0)^{n+1}$
$$= (x-x_0)^4$$

$$\Rightarrow f'(x) = 4(x-x_0)^3$$
$$f''(x) = 12(x-x_0)^2$$
$$f'''(x) = 24(x-x_0) \Rightarrow f^{(4)}(x) = 4!$$
$$\Rightarrow f^{(4)}(c) = 4!$$

- Hence, $E = K f^{(4)}(c)$

$$\boxed{E = 24K} \text{ — } (1)$$

- But $E = \int_{x_0}^{x_2} (x-x_0)^4 \, dx - \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$

$$= \frac{(x-x_0)^5}{5}\Big|_{x_0}^{x_2} - \frac{h}{3}\left[0 + 4(x_1-x_0)^4 + (x_2-x_0)^4\right]$$

$$= \frac{(x_2-x_0)^5}{5} - \frac{h}{3}\left[0 + 4h^4 + (2h)^4\right]$$

$$= \frac{(2h)^5}{5} - \frac{h}{3}\left(4h^4 + 16h^4\right)$$

$$= \frac{32h^5}{5} - \frac{20h^5}{3}$$

$$\boxed{E = \frac{-4h^5}{15}} \text{ — } (2)$$

- From (1) and (2) we get $24K = \frac{-4h^5}{15} \Rightarrow K = \frac{-h^5}{90}$

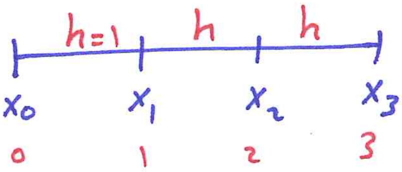
- Hence, $E = K f^{(4)}(c)$
$$= \frac{-h^5 f^{(4)}(c)}{90}$$

Exp Determine the DP of the Simpson's $\frac{3}{8}$ Rule and use it to find the truncation error.

- Recall that the Simpson's $\frac{3}{8}$ Rule is:

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + f_3\right] + \frac{-3h^5 f^{(4)}(c)}{80}$$

- It will be enough to apply Simpson's $\frac{3}{8}$ Rule over the interval $[0, 3]$ $\Rightarrow$



$$\int_0^3 f(x)\,dx = \frac{3}{8}\left[f(0) + 3f(1) + 3f(2) + f(3)\right] - \frac{3h^5 f^{(4)}(c)}{80}$$

- $\int_0^3 dx = 3 = \frac{3}{8}\left[1 + 3 + 3 + 1\right]$ with $E[1] = 0$ since $f = 1$

$\int_0^3 x\,dx = \frac{9}{2} = \frac{3}{8}\left[0 + 3 + 6 + 3\right]$ with $E[x] = 0$ since $f = x$

$\int_0^3 x^2\,dx = 9 = \frac{3}{8}\left[0 + 3 + 12 + 9\right]$ with $E[x^2] = 0$ since $f = x^2$

$\int_0^3 x^3\,dx = \frac{81}{4} = \frac{3}{8}\left[0 + 3 + 24 + 27\right]$ with $E[x^3] = 0$ since $f = x^3$

$\int_0^3 x^4\,dx = \frac{243}{5} \neq \frac{3}{8}\left[0 + 3 + 48 + 81\right] = \frac{99}{2}$ with $E[x^4] = \frac{243}{5} - \frac{99}{2} \neq 0$ since $f = x^4$

- Hence, the $DP = n = 3$ for the Simpson's $\frac{3}{8}$ Rule.
- And so, by $*$ the truncation error is

$$E = K f^{(n+1)}(c) = K f^{(4)}(c)$$

- Now to find K, we consider $f(x) = (x-x_0)^{n+1}$
$= (x-x_0)^4$

$$\Rightarrow f^{(4)}(c) = 4!$$

- Hence, $E = K f^{(4)}(c)$

$$\boxed{E = 24K} \quad (1)$$

- But $E = $ True Value − Estimate value

$$= \int_{x_0}^{x_3} (x-x_0)^4 dx - \frac{3}{8}h[f_0 + 3f_1 + 3f_2 + f_3]$$

$$= \frac{(x-x_0)^5}{5}\Big|_{x_0}^{x_3} - \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$= \frac{(x_3-x_0)^5}{5} - \frac{3h}{8}\left[0 + 3(x_1-x_0)^4 + 3(x_2-x_0)^4 + (x_3-x_0)^4\right]$$

$$= \frac{(3h)^5}{5} - \frac{3h}{8}\left[3h^4 + 3(2h)^4 + (3h)^4\right]$$

$$= \frac{243h^5}{5} - \frac{99h^5}{2}$$

$$\boxed{E = \frac{-9h^5}{10}} \quad (2)$$

- From (1) and (2) we get $24K = \frac{-9h^5}{10} \Rightarrow K = \frac{-3h^5}{80}$

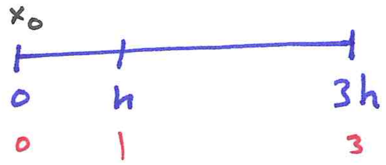
- Hence, $E = K f^{(4)}(c)$

$$= \frac{-3h^5 f^{(4)}(c)}{80}$$

Exp Given the following quadrature formula: $\boxed{152}$

$$\int_0^{3h} f(x)\,dx \approx Q[f] = \frac{3h}{4}\left[3f(h) + f(3h)\right].$$

Find its DP and its truncation error $E[f]$.



$$\int_0^3 f(x)\,dx \approx \frac{3}{4}\left[3f(1) + f(3)\right]$$

- $\int_0^3 dx = 3 = \frac{3}{4}[3+1]$ with $E[1] = 0$ since $f = 1$

$\int_0^3 x\,dx = \frac{9}{2} = \frac{3}{4}[3+3]$ with $E[x] = 0$ since $f = x$

$\int_0^3 x^2\,dx = 9 = \frac{3}{4}[3+9]$ with $E[x^2] = 0$ since $f = x^2$

$\int_0^3 x^3\,dx = \frac{81}{4} \neq \frac{3}{4}[3+27] = \frac{90}{4}$ with $E[x^3] = \frac{81}{4} - \frac{90}{4} = \frac{-9}{4} \neq 0$ since $f = x^3$

- Hence, $DP = n = 2$ and therefore, $E = K f^{(n+1)}(c) = K f'''(c)$
- Take $f(x) = x^3 \xrightarrow{\text{since } x_0 = 0} f'''(c) = 3! = 6 \Rightarrow$ so $\boxed{E = 6K}$ — ①
- But $E = \text{True} - \text{Estimate} = \int_0^{3h} x^3\,dx - \frac{3h}{4}\left[3f(h) + f(3h)\right]$

$$= \frac{x^4}{4}\Big|_0^{3h} - \frac{3h}{4}\left[3h^3 + 27h^3\right]$$

$$= \frac{81h^4}{4} - \frac{90h^4}{4}$$

$$\boxed{E = \frac{-9h^4}{4}} \quad ②$$

- From ① and ② we get

$6K = \frac{-9h^4}{4} \Rightarrow K = \frac{-3h^4}{8}$

Hence, $E = K f'''(c)$

$$= \frac{-3h^4 f'''(c)}{8}$$ ✓

Exp Given the following quadrature formula:

$$\int_{-1}^{1} f(x)\,dx \approx Q[f] = \frac{1}{2}\left[f(-1) + 3 f\left(\tfrac{1}{3}\right)\right]$$

Find its DP and its truncation error $E[f]$.

- $\int_{-1}^{1} dx = 2 = \frac{1}{2}[1+3]$ with $E[1]=0$ since $f=1$

$\int_{-1}^{1} x\,dx = 0 = \frac{1}{2}[-1+1]$ with $E[x]=0$ since $f=x$

$\int_{-1}^{1} x^2\,dx = \frac{2}{3} = \frac{1}{2}\left[1+\frac{1}{3}\right]$ with $E[x^2]=0$ since $f=x^2$

$\int_{-1}^{1} x^3\,dx = 0 \neq \frac{1}{2}\left[-1+\frac{1}{9}\right] = \frac{-4}{9}$ with $E[x^3] = 0 - \frac{-4}{9} = \frac{4}{9} \neq 0$ since $f=x^3$

- Hence, $DP = n = 2$ and therefore $E = K f^{(n+1)}(c) = K f'''(c)$
- Now Take $f(x) = (x-x_0)^{(n+1)} = (x+1)^3$ $\Rightarrow f'''(c) = 3! = 6$

$$\Rightarrow \boxed{E = 6K} \quad (1)$$

- But $E = \int_{-1}^{1} (x+1)^3\,dx - \frac{1}{2}\left[f(-1) + 3 f\left(\tfrac{1}{3}\right)\right]$

$= \left.\frac{(x+1)^4}{4}\right|_{-1}^{1} - \frac{1}{2}\left[0 + 3\left(\frac{1}{3}+1\right)^3\right]$

$= 4 - \frac{32}{9}$

$$\boxed{E = \frac{4}{9}} \quad (2)$$

- From (1) and (2) we get

$6K = \frac{4}{9}$

$$\boxed{K = \frac{2}{27}}$$

Hence, $E = K f'''(c)$

$= \frac{2 f'''(c)}{27}$

✓