Composite Trapezoidal Rule (CTR)

This method approximates the area under the curve $y = f(x)$ over $[a,b]$ using a series of trapezoids that lie above the intervals $\{[x_k, x_{k+1}]\}$.

Th (CTR)

- Assume that the interval $[a,b]$ is subdivided into M subintervals $[x_k, x_{k+1}]$ each of width $h = \frac{b-a}{M}$ using equally spaced nodes $x_k = a + kh$ for $k = 0, 1, 2, \dots, M$:

- Then, the composit trapezoidal rule is

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_M} f(x)\,dx \approx T(f,h) = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \dots + \int_{x_{M-1}}^{x_M} f(x)\,dx$$

$$= \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \dots + \frac{h}{2}(f_{M-1} + f_M)$$

$$= \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \dots + 2f_{M-1} + f_M\right)$$

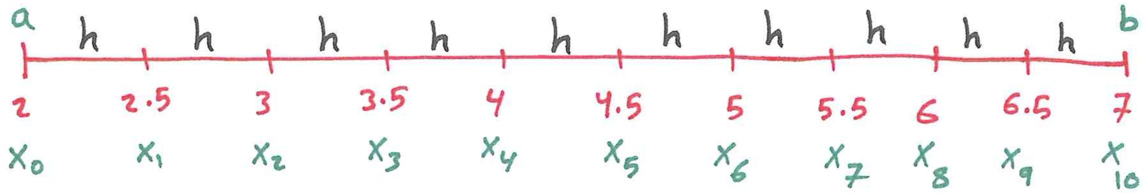
- Furthermore, the total error of $T(f,h)$ is

$$E_T(f,h) = \frac{-h^3 f''(c)}{12} \cdot M = \frac{-h^3 f''(c)}{12} \cdot \frac{b-a}{h} = \frac{-h^2 f''(c)}{12}(b-a)$$

Note that
- Number of points is $M+1$
- M is the number of subintervals = number of subintegrals = number of composition

Exp Use CTR to estimate $\int_2^7 e^x\,dx$ with 10 composition. (use 4 chopping digits).

- $h = \dfrac{b-a}{M} = \dfrac{7-2}{10} = \dfrac{5}{10} = 0.5$ and $f(x) = e^x = (2.718)^x$



- $\int_2^7 e^x\,dx \simeq T(e^x, 0.5)$

$$= \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + 2f_5 + 2f_6 + 2f_7 + 2f_8 + 2f_9 + f_{10}\right]$$

$$= \frac{0.5}{2}\left[(2.718)^2 + 2(2.718)^{2.5} + 2(2.718)^3 + \cdots + 2(2.718)^{6.5} + (2.718)^7\right]$$

$$= 0.25\left[7.387 + 2(12.17) + 2(20.07) + \cdots + 2(664.6) + 1095\right]$$

$$= 0.25\left[7.387 + 24.34 + 40.14 + 66.2 + 109.1 + 179.9 + 296.6 + 489 + 806.2 + 1329 + 1095\right]$$

$$= 0.25\,(4333)$$

$$= 1083$$

True Vaule is $\int_2^7 e^x\,dx = e^x\Big|_2^7 = e^7 - e^2$

$= 1089.2441023295$

Exp Given

x	0	2	4	6
$f(x)$	10	15	-10	8

Use CTR to estimate $\int_0^6 f(x)\,dx$

$$\int_0^6 f(x)\,dx \simeq \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + f_3\right] = \frac{2}{2}\left[10 + 2(15) + 2(-10) + 8\right] = 28$$

Exp Given

x	0	2	3	6
$f(x)$	10	15	-10	8

Use CTR to estimate $\int_0^6 f(x)\,dx$.

$$\int_0^6 f(x)\,dx \simeq \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^6 f(x)\,dx = \frac{2}{2}\left[f_0 + f_1\right] + \frac{1}{2}\left[f_1 + f_2\right] + \frac{3}{2}\left[f_2 + f_3\right]$$

$$= \left[10 + 15\right] + \frac{1}{2}\left[15 - 10\right] + \frac{3}{2}\left[-10 + 8\right] = 24.5$$

Composit Simpson Rule (CSR)

This method approximates the area under the curve $y=f(x)$ over $[a,b]$.

Th (CSR)

- Assume that the interval $[a,b]$ is subdivided into $2M$ subintervals $[x_k, x_{k+1}]$ each of width $h=\frac{b-a}{2M}$ using equally spaced nodes $x_k = a+kh$ for $k=0,1,2,\dots,2M$:

$$\underset{a=x_0}{|}\overset{h}{\text{—}}\underset{x_1}{|}\overset{h}{\text{—}}\underset{x_2}{|}\overset{h}{\text{—}}\underset{x_3}{|}\overset{h}{\text{—}}\underset{x_4}{|}\cdots\underset{x_{2M-2}}{|}\text{—}\underset{x_{2M-1}}{|}\overset{h}{\text{—}}\underset{x_{2M}=b}{|}$$

- Then, the Composite Simpson Rule is

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_{2M}} f(x)\,dx \approx S(f,h) = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots + \int_{x_{2M-2}}^{x_{2M}} f(x)\,dx$$

$$= \frac{h}{3}\left[f_0+4f_1+f_2\right] + \frac{h}{3}\left[f_2+4f_3+f_4\right] + \dots + \frac{h}{3}\left[f_{2M-2}+4f_{2M-1}+f_{2M}\right]$$

- Furthermore, the total error of $S(f,h)$ is

$$E_S(f,h) = \frac{-h^5 f^{(4)}(c)}{90}\cdot M = \frac{-h^4 f^{(4)}(c)}{180}(b-a)$$

Exp Given

x	0	1	2	4	6
$f(x)$	2	-1	3	0	10

Estimate $\int_0^6 f(x)\,dx$ using CSR.

$$\underset{0}{\bullet}\overset{h=1}{\text{—}}\underset{1}{\bullet}\overset{h=1}{\text{—}}\underset{2}{\bullet}\overset{h=2}{\text{—}}\underset{4}{\bullet}\overset{h=2}{\text{—}}\underset{6}{\bullet}$$

$$\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx$$

$$= \frac{1}{3}\left[f(0)+4f(1)+f(2)\right] + \frac{2}{3}\left[f(2)+4f(4)+f(6)\right]$$

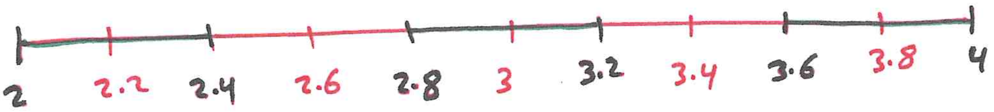
$$= \frac{1}{3}\left[2-4+3\right] + \frac{2}{3}\left[3+0+10\right]$$

$$= \frac{1}{3}+\frac{26}{3} = \frac{27}{3} = 9$$

Exp Use CSR to estimate $\int_2^4 e^x\,dx$ with 5 compositions. $\boxed{157}$

Use 4 chopping digits.

- $h = \dfrac{b-a}{2M} = \dfrac{4-2}{2(5)} = \dfrac{2}{10} = 0.2$ and $f(x) = e^x = (2.718)^x$



- $\int_2^4 e^x\,dx = \int_2^{2.4} e^x\,dx + \int_{2.4}^{2.8} e^x\,dx + \int_{2.8}^{3.2} e^x\,dx + \int_{3.2}^{3.6} e^x\,dx + \int_{3.6}^{4} e^x\,dx$

$$= \frac{0.2}{3}\left[f(2)+4f(2.2)+f(2.4)\right] + \frac{0.2}{3}\left[f(2.4)+4f(2.6)+f(2.8)\right]$$

$$+ \frac{0.2}{3}\left[f(2.8)+4f(3)+f(3.2)\right] + \frac{0.2}{3}\left[f(3.2)+4f(3.4)+f(3.6)\right]$$

$$+ \frac{0.2}{3}\left[f(3.6)+4f(3.8)+f(4)\right]$$

$$= 0.06666\Big[(2.718)^2 + 4(2.718)^{2.2} + 2(2.718)^{2.4} + 4(2.718)^{2.6} + 2(2.718)^{2.8} + 4(2.718)^3 + 2(2.718)^{3.2} + 4(2.718)^{3.4} + 2(2.718)^{3.6} + 4(2.718)^{3.8} + (2.718)^4\Big]$$

$$= 0.06666\Big[7.387 + 36.08 + 22.04 + 53.84 + 32.86 + 80.28 + 49.04 + 119.8 + 73.16 + 178.7 + 54.57\Big]$$

$$= 0.06666(707.4)$$

$$= 47.15$$

Note that the True Value is $\int_2^4 e^x\,dx = e^x\Big|_2^4 = e^4 - e^2 = 47.2090939342$

Exp Find the number of compositions and the step size needed to estimate $\int_2^7 \frac{dx}{x}$ with accuracy 5×10^{-9} using

1. CTR
2. CSR

---

1. $|E| = \left| \frac{-h^2 f''(c)}{12}(b-a) \right| \leq 5\times10^{-9}$

$a=2$, $b=7$

$f(x) = \frac{1}{x}$

$f' = -\frac{1}{x^2}$

$f'' = \frac{2}{x^3} \leq \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$

$$\left| \frac{h^2 \left(\frac{1}{4}\right)}{12} \cancel{(7-2)} \right| \leq \cancel{5}\times10^{-9}$$

$$h \leq \sqrt{12\times10^{-9}\times4} = 0.000219089$$

$$M = \frac{b-a}{h} \geq \frac{5}{0.000219089} = 22821.77562543$$

so the number of compositions is $M \geq 22822$ and # of points $= M+1$

---

2. $|E| = \left| \frac{h^4 f^{(4)}(c)}{180}(b-a) \right| \leq 5\times10^{-9}$

$f'''(x) = -\frac{6}{x^4}$

$f^{(4)} = \frac{24}{x^5} \leq \frac{24}{2^5} = \frac{24}{32} = \frac{3}{4}$

$$\frac{h^4 \left(\cancel{\frac{3}{4}}\right)(\cancel{5})}{\cancel{180}_{60}} \leq \cancel{5}\times10^{-9}$$

$$h \leq \left(240\times10^{-9}\right)^{\frac{1}{4}} = 0.022133638 4$$

$$M = \frac{b-a}{2h} \geq \frac{5}{0.0442672768} = 112.9502504206$$

Hence, $M \geq 113$