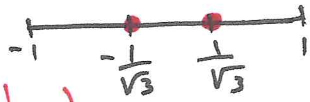
Gauss-Legendre Integration (optional)

To estimate the area under the curve $y=f(x)$, $-1 \le x \le 1 \Rightarrow$ one can use one of the following formulas:

1 Gauss-Legendre one-point Rule :

$$\int_{-1}^{1} f(x)\,dx \approx G_1(f) = 2 f_0$$

2 Gauss-Legendre two-points Rule :

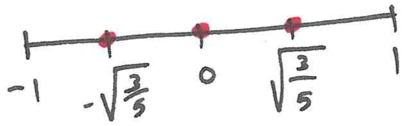


$$\int_{-1}^{1} f(x)\,dx \approx G_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Gauss-legendre two-points Rule $G_2(f)$ has Degree of Precision $\boxed{DP = 2n-1 = 3}$ and error $\boxed{E_2[f] = \frac{f^{(4)}(c)}{135}}$

3 Gauss-Legendre three-points Rule

$$\int_{-1}^{1} f(x)\,dx \approx G_3(f) = \frac{5 f\left(-\sqrt{\frac{3}{5}}\right) + 8 f(0) + 5 f\left(\sqrt{\frac{3}{5}}\right)}{9}$$



$G_3(f)$ has Degree of Precision $\boxed{DP = 2n-1 = 5}$

and error $\boxed{E_3[f] = \frac{f^{(6)}(c)}{15750}}$

Exp • Given $\int_{-1}^{1} \frac{dx}{x+2} = \ln(x+2)\Big|_{-1}^{1} = \ln(3) - \ln(1) \approx 1.09861$

• Estimate this integral using [1] $G_2(f)$ [2] $G_3(f)$ [3] $T(f,h)$ with $h=2$ [4] $S(f,h)$ with $h=1$

Use 4 chopping

[1] $$\int_{-1}^{1} \frac{dx}{x+2} \approx G_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$f(x) = \frac{1}{x+2}$

$$= f(-0.5773) + f(0.5773)$$

$$= \frac{1}{1.422} + \frac{1}{2.577} = 0.7032 + 0.3880$$

$$= 1.091$$

[2] $$\int_{-1}^{1} \frac{dx}{x+2} \approx G_3(f) = \frac{5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)}{9}$$

$$= \frac{5f(-0.7745) + 8f(0) + 5f(0.7745)}{9}$$

$$= \frac{4.081 + 4 + 1.802}{9} = \frac{9.883}{9} = 1.098$$

[3] $$\int_{-1}^{1} \frac{dx}{x+2} \approx T(f,2) = \frac{h}{2}\left[f(-1) + f(1)\right] = f(-1) + f(1)$$

$$= 1 + 0.3333 = 1.333$$

[4] $$\int_{-1}^{1} \frac{dx}{x+2} \approx S(f,1) = \frac{h}{3}\left[f(-1) + 4f(0) + f(1)\right]$$

$$= 0.3333\left[1 + 2 + 0.3333\right] = 0.3333(3.333) = 1.110$$

Exp Estimate $\int_{-1}^{1} \sin x \, dx$ using [1] $G_2(f)$ [2] $G_3(f)$

1) $$\int_{-1}^{1} \sin x \, dx \approx G_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = -\sin\frac{1}{\sqrt{3}} + \sin\frac{1}{\sqrt{3}} = 0$$

2) $$\int_{-1}^{1} \sin x \, dx \approx G_3(f) = \frac{5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)}{9} = 0$$

Gauss - Legendre Translation

- How to use $G_2(f)$ and $G_3(f)$ to estimate $\int_a^b f(x)\,dx$
- we use change of variables to transform the limits of integration from $[a,b]$ to $[-1,1]$:

$$x = \frac{a+b}{2} + \frac{b-a}{2}\,t \quad \Rightarrow \quad dx = \frac{b-a}{2}\,dt$$

- when $t=-1 \Rightarrow x = \frac{a+b}{2} - \frac{b-a}{2} = a$

$t = 1 \Rightarrow x = \frac{a+b}{2} + \frac{b-a}{2} = b$

- Hence, $\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}t\right)\frac{b-a}{2}\,dt$

$$= \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}t\right)dt$$

$\Downarrow$

$$G_2(f) = \frac{b-a}{2}\left[f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(\frac{-1}{\sqrt{3}}\right)\right) + f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right)\right)\right]$$

and

$$G_3(f) = \frac{b-a}{2}\left[\frac{5f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(-\sqrt{\frac{3}{5}}\right)\right) + 8f\left(\frac{a+b}{2}\right) + 5f\left(\frac{a+b}{2} + \frac{b-a}{2}\left(\sqrt{\frac{3}{5}}\right)\right)}{9}\right]$$

Exp Use two-points Gauss-Legendre rule to approximate $\boxed{162}$

$$\int_1^5 e^{-x}\,dx = -e^{-x}\Big|_1^5 = -e^{-5} + e^{-1} = 0.3746173882$$

- $x = \frac{a+b}{2} + \frac{b-a}{2}t = 3 + 2t$

$dx = 2\,dt$

- $$\int_1^5 e^{-x}\,dx = \int_{-1}^{1} e^{-(3+2t)}\,2\,dt = 2\left[f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= 2\left[e^{-(3+\frac{-2}{\sqrt{3}})} + e^{-(3+\frac{2}{\sqrt{3}})}\right] = 0.3473369892$$

Exp Use three-points Gauss-Legendre rule to estimate

$$\int_1^5 \frac{dx}{x} = \ln x\Big|_1^5 = \ln 5 = 1.6094379124.$$ Use 4 chopping digits.

- $x = \frac{a+b}{2} + \frac{b-a}{2}t = 3 + 2t$ with $dx = 2\,dt$
- Now ➜ $\int_1^5 \frac{dx}{x} = \int_{-1}^{1} \frac{2\,dt}{3+2t}$ with $f(t) = \frac{2}{3+2t}$

$$= \frac{5f(-\sqrt{\frac{3}{5}}) + 8f(0) + 5f(\sqrt{\frac{3}{5}})}{9}$$

$$= \frac{1}{9}\left[5f(\bar{0}.7745) + 8f(0) + 5f(0.7745)\right]$$

$$= 0.1111\left[5(1.378) + 8(0.6666) + 5(0.4396)\right]$$

$$= 0.1111\left[6.890 + 5.332 + 2.198\right]$$

$$= 0.1111(14.41)$$

$$= 1.6$$