Ch 9 : Numerical Approximation for IVP

- Given the IVP:

$$\frac{dy}{dt} = f(t, y(t)) \quad \text{with } y(t_0) = y_0$$

- To estimate the solution $y(t)$ numerically:
  - ⇒ we find the values of y at different values of t
  - ⇒ Then Approximate y using interpolation

Exp Consider the IVP : $y' = t + y \sin t^2$, $y(2) = -1$

- Clearly $t_0 = 2$ and $y_0 = -1$
- Write $y' = f(t, y)$
- Find $f'(t, y)$ by differentiating w.r.t t

$$f(t,y) = t + y \sin t^2$$

$$f'(t) = 1 + y\left[\cos t^2 \; (2t)\right] + \sin t^2 \; y'$$

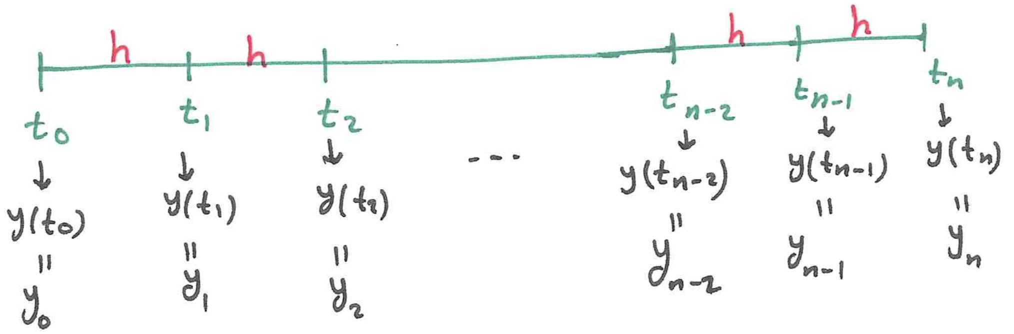
$$= 1 + 2ty \cos t^2 + \sin t^2 \left(t + y \sin t^2\right)$$

General Principle:

out • Given an IVP: $y' = f(t,y)$, $y(t_0) = y_0$

out • To estimate the values of y on $[a,b] = [t_0, t_n]$:

$\Rightarrow$ Find $h = \dfrac{t_n - t_0}{n} = \dfrac{b-a}{n}$



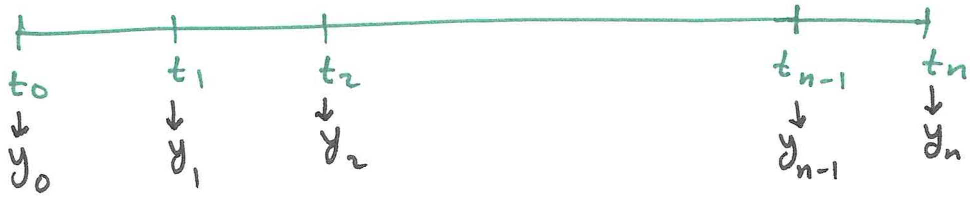
$\Rightarrow$ Find $y(t_k) = y_k$ where $k = 0, 1, \ldots, n$
and $t_k = t_0 + kh$

• We can find $y_1, y_2, \ldots, y_n$ using

1. Euler's Method
2. Taylor's Method of order 2
3. Heun's Method
4. Runge-Kutta of order 4 (RK4)

1 Euler's Method

- Given an IVP : $y' = f(t, y)$, $y(t_0) = y_0$
- $h = \dfrac{t_n - t_0}{n} = \dfrac{b-a}{n}$



- Take Taylor's expansion for $y(t)$ about t_0

$$y(t) \simeq y(t_0) + (t - t_0)\, y'(t_0)$$
$$= y_0 + (t - t_0)\, f(t_0, y_0)$$
$$y(t_1) = y_0 + (t_1 - t_0)\, f(t_0, y_0)$$
$$\boxed{y_1 = y_0 + h\, f(t_0, y_0)}$$

- Similarly, take Taylor's expansion for $y(t)$ about t_1:

$$y(t) = y(t_1) + (t - t_1)\, y'(t_1)$$
$$y(t_2) = y_1 + (t_2 - t_1)\, f(t_1, y_1)$$
$$\boxed{y_2 = y_1 + h\, f(t_1, y_1)}$$

$$\vdots$$

$$\boxed{y_n = y_{n-1} + h\, f(t_{n-1}, y_{n-1})}$$

In general: The Euler's Method is

$$y_{k+1} \approx y_k + h f(t_k, y_k) \quad , \quad k = 0, 1, \dots, n$$

Exp Given IVP: $y' = \dfrac{t-y}{2}$, $y(0.1) = 2$

Estimate $y(0.7)$ using Euler's Method with step size $h = 0.2$

0.1	0.3	0.5	0.7
t_0	t_1	t_2	t_3
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$y_0 = 2$	y_1	y_2	y_3

$$f(t, y) = \frac{t-y}{2}$$

$$f(0.1, 2) = \frac{0.1 - 2}{2} = -0.95$$

$$y_1 = y_0 + h f(t_0, y_0)$$
$$= 2 + 0.2 f(0.1, 2)$$
$$= 2 + 0.2(-0.95)$$
$$= 1.81$$

$$f(0.3, 1.81) = \frac{0.3 - 1.81}{2} = -0.755$$

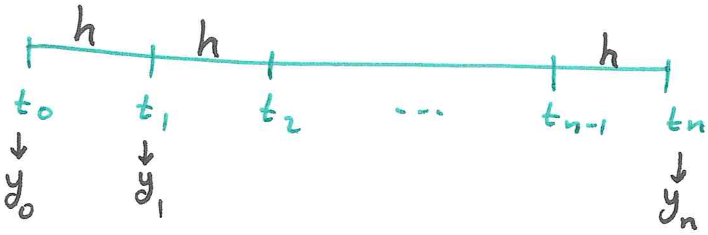
$$y_2 = y(0.5) = y_1 + h f(t_1, y_1)$$
$$= 1.81 + 0.2 f(0.3, 1.81) = 1.81 + 0.2(-0.755)$$
$$= 1.659$$

$$y_3 = y(0.7) = y_2 + h f(t_2, y_2) = 1.659 + 0.2 f(0.5, 1.659)$$
$$= 1.659 + 0.2\left(\frac{0.5 - 1.659}{2}\right)$$
$$= 1.5431$$

2 Taylor's Method of order 2

- Given an IVP: $y' = f(t,y)$, $y(t_0) = y_0$

  with $[a,b] = [t_0, t_n]$, $h = \frac{b-a}{n}$



- Take Taylor's expansion of order 2 for $y(t)$ about t_0 :

$$y \approx y(t_0) + (t-t_0)\, y'(t_0) + \frac{(t-t_0)^2}{2} y''(t_0)$$

$$\boxed{y_1 = y_0 + h f(t_0, y_0) + \frac{h^2}{2} f'(t_0, y_0)}$$

- Similarly: $y_2 = y_1 + h f(t_1, y_1) + \frac{h^2}{2} f'(t_1, y_1)$

$$\vdots$$

In general $\Rightarrow$

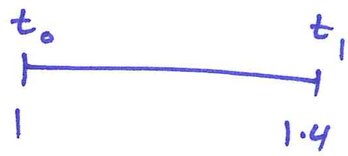
$$\boxed{y_{k+1} = y_k + h f(t_k, y_k) + \frac{h^2}{2} f'(t_k, y_k)}$$

for $k = 0, 1, 2, \dots, n-1$

Exp Given the IVP: $y' = \frac{t-y}{2}$, $y(1) = 2$

Estimate $y(1.4)$ using Taylor's Method of order 2 with step size $h = 0.4$

$$y_1 = y_0 + h f(t_0, y_0) + \frac{h^2}{2} f'(t_0, y_0)$$



$$= 2 + 0.4\, f(1,2) + \frac{(0.4)^2}{2} f'(1,2)$$

$$= 2 + 0.4 \left(\frac{1-2}{2}\right) + 0.08 \left(\frac{1}{2} - \frac{1-2}{4}\right)$$

$$= 2 - 0.2 + 0.04 + 0.02$$

$$= 1.86$$

$t_0 = 1$, $y_0 = 2$

$f(t,y) = \frac{t-y}{2}$

$f'(t,y) = \frac{1}{2}(1 - y')$

$= \frac{1}{2} - \frac{t-y}{4}$

3 Heun's Method (Modification of Euler's Method)

- Given an IVP: $y' = f(t,y)$, $y(t_0) = y_0$
- Integrate both sides $\Rightarrow$

$$\int_{t_0}^{t_1} y'(t)\,dt = \int_{t_0}^{t_1} f(t,y)\,dt$$

$$y(t)\Big|_{t_0}^{t_1} = \frac{h}{2}\left[f(t_0,y_0) + f(t_1,y_1)\right]$$

$\Downarrow$ From Euler's Method $\Downarrow$

$$y(t_1) - y(t_0) = \frac{h}{2}\left[f(t_0,y_0) + f\left(t_1, y_0 + hf(t_0,y_0)\right)\right]$$

$$y_1 = y_0 + \frac{h}{2}\left[f(t_0,y_0) + f\left(t_1, y_0 + hf(t_0,y_0)\right)\right]$$

- Similarly $\Rightarrow$

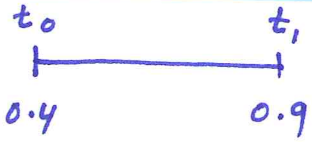
$$y_2 = y_1 + \frac{h}{2}\left[f(t_1,y_1) + f\left(t_2, y_1 + hf(t_1,y_1)\right)\right]$$

$\vdots$

Exp Given the IVP: $t^2 + y' = (1+e^t)y^3$, $y(0.4)=1$

Estimate $y(0.9)$ using Heun's Method with step size $h=0.5$ and use 4 chopping digits.

- $h = 0.5$, $t_0 = 0.4$, $y_0 = 1$



- $f(t,y) = (1+e^t)y^3 - t^2$

$$= \left[1 + (2.718)^t\right] y^3 - t^2$$

- $f(t_0,y_0) = f(0.4,1) = \left(1 + (2.718)^{0.4}\right)(1)^3 - (0.4)^2$

$$= 1 + 1.491 - 0.16$$

$$= 2.331$$

- $f(t_1, y_0 + hf(t_0,y_0)) = f\left(0.9, 1 + 0.5(2.331)\right) = f(0.9, 2.165)$

$$= \left[1 + (2.718)^{0.9}\right](2.165)^3 - (0.9)^2$$

$$= [1 + 2.459](10.14) - 0.81$$

$$= 35.07 - 0.81$$

$$= 34.26$$

- Hence, $y_1 = y(0.9) = y_0 + \frac{h}{2}\left[f(t_0,y_0) + f\left(t_1, y_0 + hf(t_0,y_0)\right)\right]$

$$= 1 + \frac{0.5}{2}\left[2.331 + 34.26\right]$$

$$= 1 + 0.25(36.59)$$

$$= 1 + 9.147$$

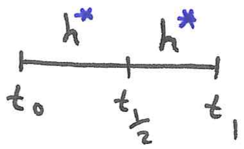
$$= 10.14$$

4 Runge-Kutta Method of order 4 (RK4)

- Given an IVP: $y' = f(t,y)$, $y(t_0) = y_0$

$$\int_{t_0}^{t_1} y'(t)\, dt = \int_{t_0}^{t_1} f(t,y)\, dt \quad \cdots \; *$$

- Apply Simpson's Rule $\int_{t_0}^{t_1} f(t)\,dt \approx \frac{h^*}{3}\left(f_0 + 4 f_{\frac{1}{2}} + f_1\right)$



with $h^* = \frac{h}{2} \Rightarrow *$ becomes

$$\$ \;\cdots \quad y(t_1) - y(t_0) = \frac{h}{6}\left[f(t_0, y_0) + 4 f(t_{\frac{1}{2}}, y_{\frac{1}{2}}) + f(t_1, y_1)\right]$$

where $t_{\frac{1}{2}}$ is the midpoint of the interval,

- Take $f_1 = f(t_0, y_0)$,

$f_4 = f(t_1, y_1)$,

$\frac{f_2 + f_3}{2} \approx f(t_{\frac{1}{2}}, y_{\frac{1}{2}})$ " average of f_2 and f_3 "

and substitute them in \$ $\Rightarrow$

$$y_1 = y_0 + \frac{h}{6}\left[f_1 + 2(f_2 + f_3) + f_4\right]$$

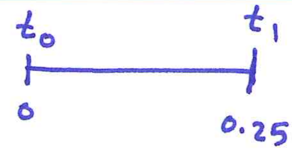
In general: $y_{k+1} = y_k + \frac{h}{6}\left[f_1 + 2f_2 + 2f_3 + f_4\right]$ where

$$f_1 = f(t_k, y_k)$$
$$f_2 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_1\right)$$
$$f_3 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_2\right)$$
$$f_4 = f\left(t_k + h, y_k + h f_3\right)$$

$k = 0, 1, 2, \ldots, n-1$

Exp Given the IVP: $y' = \frac{t-y}{2}$, $y(0) = 1$
Estimate $y(0.25)$ using RK4 with step size $h = \frac{1}{4}$
and use 4 chopping digits

- $t_0 = 0$, $y_0 = 1$, $h = 0.25$ and



$$f(t,y) = \frac{t-y}{2}$$

- $f_1 = f(t_0, y_0) = f(0,1) = \frac{0-1}{2} = -0.5$

- $f_2 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_1\right) = f\left(0 + \frac{0.25}{2}, 1 + \frac{0.25}{2}(-0.5)\right)$

$$= f(0.125, 0.9375) = \frac{0.125 - 0.9375}{2} = -0.4062$$

- $f_3 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_2\right) = f\left(0.125, 1 + 0.125(-0.4062)\right)$

$$= f(0.125, 0.9492) = \frac{0.125 - 0.9492}{2} = -0.4121$$

- $f_4 = f(t_0 + h, y_0 + h f_3) = f\left(0 + 0.25, 1 + 0.25(-0.4121)\right)$

$$= f(0.25, 0.897) = \frac{0.25 - 0.897}{2} = -0.3235$$

- Hence, $y_1 = y_0 + \frac{h}{6}\left[f_1 + 2f_2 + 2f_3 + f_4\right]$

$$= 1 + \frac{0.25}{6}\left[-0.5 - 2(0.4062) - 2(0.4121) - 0.3235\right]$$

$$= 1 + 0.04166\left[-0.5 - 0.8124 - 0.8242 - 0.3235\right]$$

$$= 1 - 0.1024$$

$$= 0.8976$$