MATH 330

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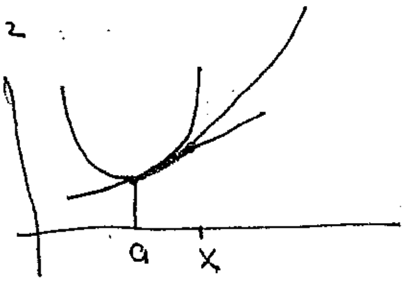
ylor Theorem :-

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{n}(a)}{n!}(x-a)^n + \cdots$$



$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$f(x) \simeq f(a) + f'(a)(x-a)$ linear estimation.

$$\text{Error} = \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$\text{Error} = \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

in general

$$f(x) \approx f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$\text{Error} = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \cdots$ (infinite Terms).

Taylor :-

$\text{Error} = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ c between a, x

صار Error حد واحد
لكن المشكلة كيف نجد C

$$|\text{Error}| \leq \max_{\leq x \leq} \frac{|f^{(n+1)}(x)|(x-a)^{n+1}}{(n+1)!}$$

$$\Rightarrow \boxed{f(x) = \underbrace{f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{E_n(x)\ \text{Error}}}$$

- e^x, $a = 0$

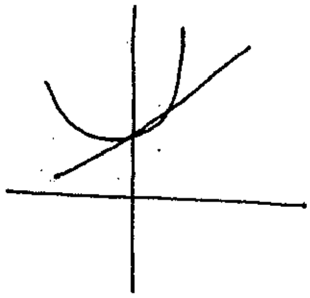
$$e^x = f(0) + f'(0)(x-0) + \frac{f''(c)}{2!}(x-0)^2$$

$$e^x = 1 + x + \frac{e^c x^2}{2!}$$

$e^x \approx 1 + x$ with error $\frac{e^c x^2}{2!}$

$$e^x = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(c)(x-0)^3}{3!}$$

$$e^x \approx 1 + x + \frac{x^2}{2} \qquad \text{error} = \frac{e^c x^3}{6}$$



$$e^{0.1} \approx 1 + 0.1 + \frac{0.01}{2} \qquad \text{error } \frac{e^c(0.001)}{6} < 1 \times 10^{-3}$$

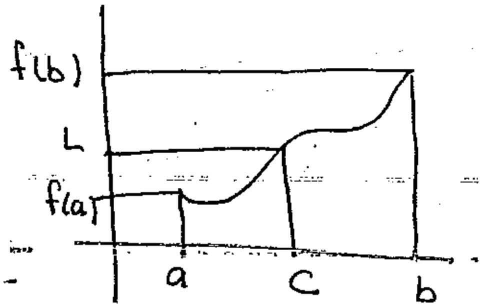
$$\approx 1.105$$

$$c \in [0, 0.1]$$

upper bound for error $\frac{e^c(0.001)}{6} \leq \frac{\overset{e^1}{3}(0.001)}{6} \leq 0.0005$.

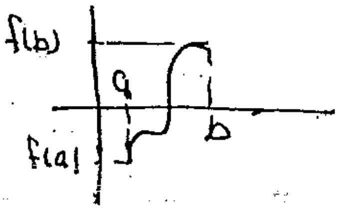
intermediate Valid Theorem (IVT)

- $f(x)$ is continous
- L between $f(a)$ and $f(b)$
- Then $\exists\, c \in (a,b)$ such that $f(c) = L$



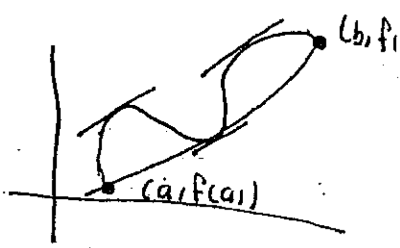
belzano

- $f(x)$ is continous
- $f(a) = f(b) < 0$
- Then $\exists\, c \in (a,b)$ such that $f(c) = 0$



mean value theorem (MVT)

- $f(x)$ is continous on $[a,b]$
- $f(x)$ is differentiable on (a,b)
- then $\exists\, c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$



sis

suppose that p^n is an approximation to P

the error is $E_p = P - p^n$

the relative error $R_p = \frac{E_p}{P} = \frac{P - p^n}{P}$

Ex:- 1. let $x = 3.141592$

$x^n = 3.14$ ٣ منازل متأكدين منهم

$E_x = 3.141592 - 3.14 = 0.001592$

$R_x = \frac{0.001592}{3.141592} = 0.000507$.

2. let $y = 1,000,000$

$\hat{y} = 999,996$ ٥ منازل متأكدين منهم الخطأ في ٤

$E_y = 4$

$R_y = \frac{4}{1,000,000} = 4 * 10^{-6}$

3. let $z = 0.000,012$

$\hat{z} = 0.000,009$ مش متأكدين ولا من اي منزلة

$E_z = 0.000,003$

$R_z = 0.25$

normalized decimal Form:-

$\pm\, 0.d_1 d_2 d_3 \cdots \times 10^n$

$d_1 \neq 0$

$x^2 = 2$

$x^2 - 2 = 0$ حسب بلزانو

$0 = \frac{1+2}{2} = 1.5$

$1 = \frac{1+1.5}{2} = 1.25$

$2 = \frac{1+1.25}{2} = 1.125 = 0.1125 * 10^1$

2 signifigant digits $\Rightarrow$ Error $\leq 10^{-2}$

بعد اول منزلة غير صفرية

Def:- the number $\hat{P}$ is said to approximate P to d significant digits if d is the largest positive integer for which

$$\frac{|P-\hat{P}|}{|P|} < \frac{10^{-d}}{2}$$

i,e $2|R_e| \lessapprox 10^{-d}$

ex:-

1. $x = 3.141592$

$x^n = 3.14$

$E_x = 3.141592 - 3.14 = 0.001592$

$R_x = \frac{0.001592}{3.141592} = 0.000507$

$2|R_x| = 0.001014 \approx 10^{-3}$
$< 10^{-4}$

المثال السابق 2. $2|R_y| = 8\times10^{-6} < 10^{-3}$

10^{-2}

10^{-3}

10^{-4}

10^{-5}

10^{-6}

3. $2|R_z| = 0.5 \nless 10^{-1}$

no significant bits.

- if $P = \pm 0.d_1d_2 \cdots d_nd_{n+1} \cdots \times 10^n$ is the normalized decimal form of the number P, $d_1 \neq 0$, then the k^{th} digit chopped floating point representation of P is

$$fl_{chop}(P) = \pm 0.d_1d_2 \cdots d_k * 10^n$$

the k^{th} digit round off Floating point representation of p is

$$fl_{round}(P) = \pm 0.d_1d_2 \cdots d_{k-1} r_k \times 10^n$$

where r_k is obtained by rounding $d_k, d_{k+1}, d_{k+2} \cdots$

- $P = 0.1234|444445$

4 digits Chopped

$fl_{chop}(P) = 0.1234$

$fl(p)_{round} = 0.1235$

<u>Final</u> مهم

Use 4 digits arthmatic (round) ٤ منازل بعد اول منزلة غير صفرية

$$\frac{\frac{3}{7}+\frac{5}{8}+\left(\frac{11}{15}\right)}{21} = ?? \quad \text{or} \quad \frac{\frac{3}{7}+0.5967+\frac{11}{15}}{21} = ??$$

$$\frac{(0.4286+0.5967)+0.7333}{21}$$

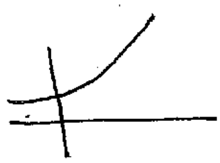
$$0.4286+0.5967 = 1.0253 \approx 1.025$$

$$1.025+0.7333 = 1.7583 \approx 1.758$$

$$\frac{1.758}{21} = 0.08371$$

- <u>order of estimation</u>

$e^x \cong 1+x$



$e^x = 1+x+\frac{x^2}{2!} + \frac{x^3}{3!}+\cdots$

$e^x \approx 1+x$

- $e^h \approx 1+h \qquad h \simeq 0$ order of approximation.

  Error $= \frac{h^2}{2!} \approx O(h^2)$

$e^{0.1} \approx 1+0.1 \approx 1.1 \qquad \text{error} = \overset{\text{const}}{C} h^2$

$e^{0.1} = 1.105170918 \qquad = C(0.1)^2$

$= C(0.01)$

$\leq 10^{-2}$

- $e^h = 1+h+\frac{h^2}{2!}$

  Error $= Ch^3 = O(h^3)$

  $e^{0.1} = 1+0.1+\frac{0.01}{2}$

  $= 1.105$

  Error $\approx C(0.1)^3$

  $\approx C(0.001) \leq 10^{-3}$

$\sin(0.1) \approx 0.1$

- $\sin h \approx h$ with error $O(h^3)$.

  $\sin h \approx h - \frac{h^3}{3!}$ with error $O(h^5)$

suppose $e^h \simeq 1+h$ Error $= O(h^2)$ (0.01)

$\sin h = h - \frac{h^3}{3!}$ Error $= O(h^5)$ (0.00001)

$$e^h + \sin h \approx 1 + 2h - \frac{h^3}{3!} \quad \text{with Error } O(h^2) + O(h^5)$$

لاتؤثر

الاصغر هي التي تؤثر

$$\approx 1 + 2h + O(h^2)$$

<u>def:-</u> order of approximation

assum that $f(h)$ is approximated by $p(h)$ and there exists a real Constant $\underline{M \geq 0}$ and a positive integer n so that

$$\frac{|f(h) - p(h)|}{|h^n|} \leq M \quad \text{for small } h$$

we say $p(h)$ approximate $f(h)$ with order of approximation $O(h^n)$ and we write $f(h) = p(h) + O(h^n)$

$$|f(h) - p(h)| \leq M|h^n|$$

$$f(h) - p(h) \approx C h^n$$

<u>Ex:-</u> Show that $p(h) = 1+h$ estimate of $f(h) = e^h$ with order $O(h^2)$

or

show that $e^h = 1 + h + O(h^2)$

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots$$

$$\frac{|e^h - (1+h)|}{|h^2|} = \frac{\frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots}{h^2} = \frac{1}{2} + \frac{h}{3!} + \frac{h^2}{4!} + \frac{h^3}{5!} + \cdots$$

$\not<$

harmonic Series $\left(\sum \frac{1}{n}\right)$ divargens

$$< \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

$$< \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \rightarrow$$

geomatric series $= \frac{1/2}{1 - 1/2}$

$= 1$

$$e^h = 1 + h + O(h^2)$$

exercise

show that

- $\sin h = h - \frac{h^3}{3!} + O(h^5)$

2- $f(h) = \sum_{k=0}^{n} f^{k}(h)\, x^k + O(h^{n+1})$

Theory:- نظرية

assume that $f(h) = P(h) + O(h^n)$

$g(h) = q(h) + O(h^m)$

and $r = \min[m,n]$

then

$f(h) \pm g(h) = p(h) + q(h) + O(h^r)$

$f(h) \cdot g(h) = p(h)\, q(h) + O(h^r)$

$\frac{f(h)}{g(h)} = \frac{p(h)}{q(h)} + O(h^r) \qquad q(h), g(h) \neq 0.$

Ex:-

$f(h) = p(h) + O(h^3)$

$g(h) = q(h) + O(h^2)$

$\frac{f(h)}{g(h)} = \frac{p(h)}{q(h)} + O(h^2)$

Ex:- (loss of significant)

$f(x) = x(\sqrt{x+1} + \sqrt{x})$

$g(x) = \frac{x}{\sqrt{x+1} + \sqrt{x}}$

Use 6 digits arthmatic and round to find $f(500)$, $g(500)$

$f(500) = 500(\sqrt{501} - \sqrt{500})$

$= 500(22.3830 - 22.3607)$

$= 500(0.0223000) = 11.1500.$

$$g(500) = \frac{500}{\sqrt{501} + \sqrt{500}} = \frac{500}{22.3830 + 22.3607} = 11.1748$$

exact answer $= 11.174755\ldots$

الطريقة الثانية أحسن لأن في الحالة الأولى
عملية الطرح ضرتنا signifigant digits

$$\frac{3}{17} = 0.176470588 + \epsilon$$

Note:-

$$P = \tilde{P} + \epsilon_P$$
$$q = \tilde{q} + \epsilon_q$$
$$P + q = \tilde{P} + \tilde{q} + \epsilon_P + \epsilon_q$$
$$= \tilde{P} + \tilde{q} + \epsilon_{P+q}$$
$$P \cdot q = (\tilde{P} + \epsilon_P)(\tilde{q} + \epsilon_q)$$
$$= \tilde{P}\tilde{q} + \tilde{P}\epsilon_q + \tilde{q}\epsilon_P + \epsilon_P \epsilon_q$$
$$= \tilde{P}\tilde{q} + \epsilon_{Pq}$$

- $P = 9.8 \times 10^{6} + 35 \times 10^{-9}$
  $\tilde{q} = 3.6 \times 10^{7} + 2.4 \times 10^{-9}$

$[a_0, b_0] = [a, b]$

$$c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$$

$f(c_0)$

if $f(c_0) = 0$ done.

else if $f(c_0) \cdot f(a_0) < 0 \Rightarrow [a_1, b_1] = [a_0, c_0]$

else $[a_1, b_1] = [c_0, b_0]$

$$c_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$$

- example

Solve $x \sin x = 1$.

$f(x) = x \sin x - 1$.

$f(0) = -1$.

$f(2) = 0.81859485$.

$$c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$$

$$= 2 - \frac{0.81859485(2-0)}{0.81859485 - (-1)} = 1.09975017.$$

$F(c_0) = 1.09975017 \sin(1.09975017) - 1$

$= -0.02001912$.

$[a_1, b_1] = [1.09975017, 2]$.

$$c_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)} = 2 - \frac{0.81859485(2 - 1.09975017)}{0.81859485 - (-0.02001912)}$$

$= 1.12124074$

$f(c_1) = 0.00983461$

$[a_2, b_2] = [1.09975017, 1.12124074]$

ثبتنا ٥ خا $c_2 = 1.11416120$

$c_3 = 1.11415714$

Section 2.1

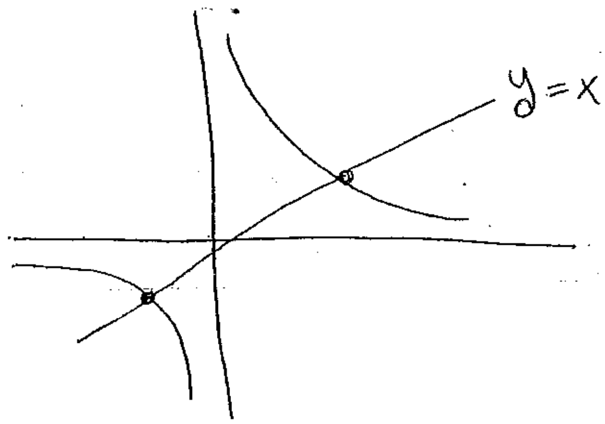
Fixed point itteration

- To solve $f(x)=0$ we solve $x=g(x)$ [where $f(x)=x-g(x)$]
  $\downarrow$ [Fixed point].

i.e to Find the roots of $f \to$ we Find the Fixed point of $g(x)$.

Def:- p is a fixed point of g iff $g(p)=p$.

1. $g(x)=\frac{1}{x}$.
   Fixed points $1, -1$.
   $g(p)=p$.
   $\frac{1}{p}=p \Rightarrow p^2=1 \Rightarrow p=\pm 1$.
2. $g(x)=x+1$. No Fixed points.
3. $g(x)=x$. all points are Fixed points.



Def:- Fixed point iteration:-

start with P_0, $P_{n+1}=g(P_n)$, $n=0,1,2,3,\ldots$

$P_1=g(P_0)$
$P_2=g(P_1)$
$\vdots$

Theorem:-

if the Fixed point itteration gonverges to P, then P is the Fixed point of g.

$\lim P_n = P \Rightarrow \lim_{n\to\infty} P_{n+1} = \lim_{n\to\infty} g(P_n) = g(\lim_{n\to\infty} P_n) = g(P)$

since $P_{n+1} = g(P_n)$ $\quad = P$.

example:-

Solve $x^2 - 2x - 3 = 0 \Rightarrow f(x) = 0$.

$(x-3)(x+1) = 0$.

$x = 3$.
$x = -1$.

$x^2 = 2x + 3$.

$x = \sqrt{2x+3} = g(x)$.

if $P_0 = 4$.

$P_1 = g(4) = g(P_0) = \sqrt{11} = 3.31662$.

$P_2 = g(P_1) = g(3.31662) = \sqrt{9.63325} = 3.10375$

$P_3 = 3.03439$

$P_4 = 3.01184$

• Note that 3 is a fixed point of $g(x) = \sqrt{2x+3}$ because $g(3) = 3$

$P_n \to 3$

way2:- x divargance.

$2x = x^2 - 3$.

$x = \frac{x^2-3}{2} = g(x)$.

$P_0 = 4$

$P_1 = g(4) = 6.5$.

$P_2 = g(6.5) = 19.625$.

$P_3 = 191.07$.

way3:-

$x(x-2) = 3 \Rightarrow x = \frac{3}{x-2} = g(x)$.

$P_0 = 4$

$P_1 = g(4) = \frac{3}{2} = 1.5$.

$P_2 = -6$

$P_3 = -0.375$

$P_4 = -1.26315$.

$P_5 = -0.919355$

$P_6 = -1.02762$

$P_7 = -0.990876$

Theorem:- (fixed poind Theorem I).

assume $g \in C[a,b]$ if $g(x) \in [a,b]$ for all $x \in [a,b]$ then g has a Fixed point in $[a,b]$ Furthermore if $|g'(x)| \leq k < 1$ for all $x \in (a,b)$ then g has a unique Fixed point.

Proof:-

if $g(a) = a$ or $g(b) = b$ Done.

if not $g(a) > a$ and $g(b) < b$

let $h(x) = g(x) - x$, h continous.

$h(a) = g(a) - a > 0$

$h(b) = g(b) - b < 0$

by belzano $\exists c \in \mathbb{C}$ shuch that $h(c) = 0$.

$g(c) - c = 0$

$\boxed{g(c) = c}$

Uniqueness

Suppose $\exists P_1, P_2$ such that $g(P_1) = P_1$, $g(P_2) = P_2$

Using mean Value theorem on (P_1, P_2)

$\exists c \in (P_1, P_2)$ such that $\left| \frac{g(P_2) - g(P_1)}{P_2 - P_1} \right| = |g'(c)| < 1$

$\frac{P_2 - P_1}{P_2 - P_1} = 1 \Rightarrow 1 < 1 \rightarrow$ تناقض Contradict

$P_1 = P_2$. $\therefore$ X

theorem:- (fixed point iteration theorem)

assume that $g(x)$ and $g'(x)$ are continous on a balanced interval $(a,b) = (P-\delta, P+\delta)$ that contains a Unique Fixed point P and that the started value P_0 is chosen in this interval.

1. if $g'(x) \leq k < 1$ for all $x \in (a,b)$ then the FPI convarge. $P_{n+1} = g(P_n)$ will convarge (attractive Fixed point).
2. if $g'(x) > 1$ for all $x \in (a,b)$ then the Fixed point itteration divarges (we call it repolsive Fixed point).

Note :-

if p is given we can replace the above two conditions by

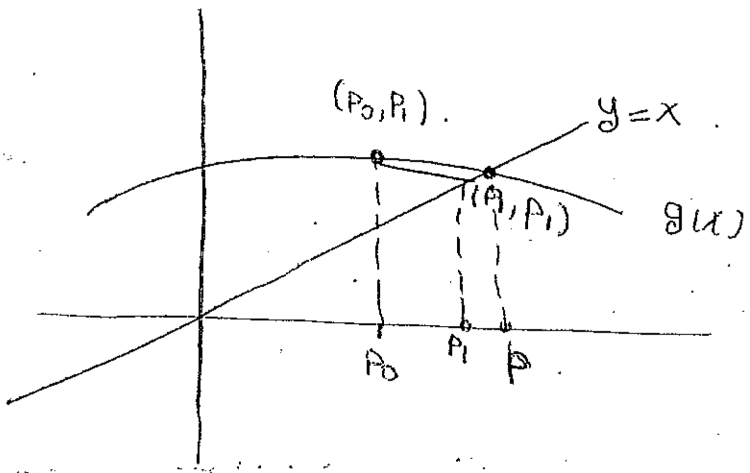
1. if $|g'(p)| < 1 \rightarrow$ the FPI converges.
2. if $|g'(p)| \geq 1 \rightarrow$ the FPI divarges.

<u>convargence</u> $\quad |g'(x)| < 1$.

$-1 < g'(x) < 0 \qquad\qquad 0 < g'(x) < 1$.

$0 < g'(x) < 1$

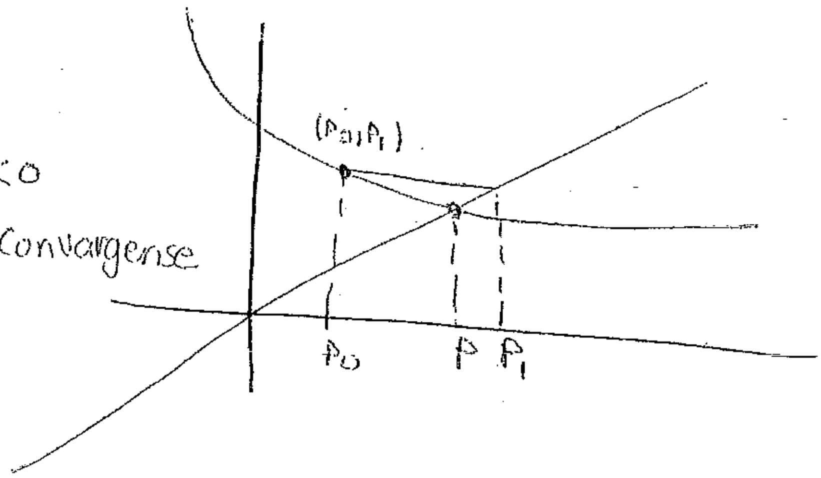
monotone convargense.



$P_1 = g(P_0)$

$-1 < g'(x) < 0$

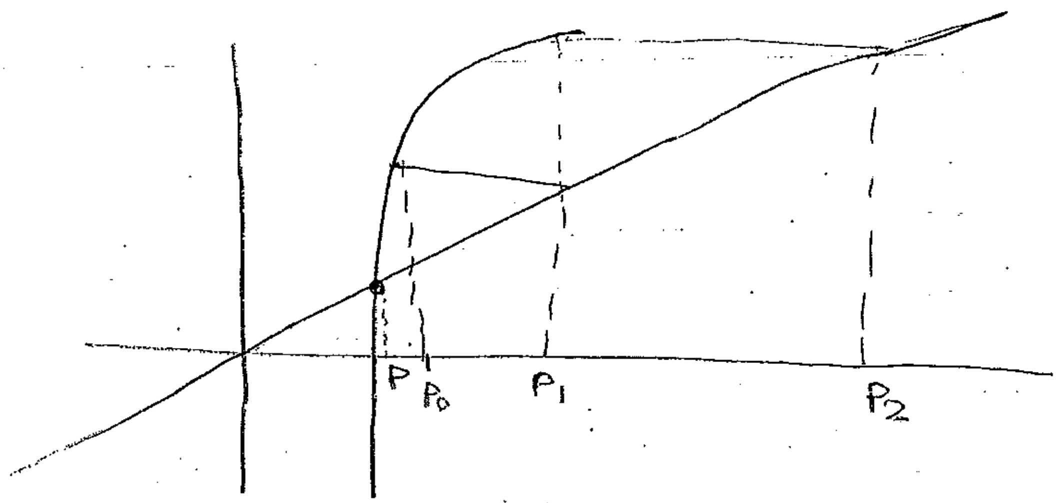
alternating convargense



divergence $|g'(x)| > 1$

$g'(x) > 1$ $\qquad$ $g'(x) < -1$.

$g'(x) > 1$



<u>example</u>

invastigate the nature of the FPI and show your answer by examples for

$$g(x) = 1 + x - \frac{x^2}{4}$$

<u>Solution</u>

$x = g(x)$.

$\cancel{x} = 1 + \cancel{x} - \frac{x^2}{4}$.

$x^2 = 4$

$x = \pm 2$ (Fixed points)

when $x = 2$.

$g'(x) = 1 - \frac{x}{2}$.

$|g'(2)| = 0 < 1 \Rightarrow$ Convargance Fixed point. (attractive Fixed point)

to show that :-

let $P_0 = 1.6$.

$P_1 = g(1.6) = 1.96$.

$P_2 = g(1.96) = 1.996$.

$P_n \rightarrow 2$.

if $P_0 = 2.5$.

$P_1 = g(2.5)$

$\longrightarrow P_n \rightarrow 2$.

at $x = -2$

$|g'(-2)| = 2 > 1$ divarge $\rightarrow$ FPI divarge. (Repulsive Fixed point).

$P_0 = -2.05$.

$P_1 = g(-2.05) = -2.1 \; ---$

$P_2 = g(-2.1) = -2.2$.

$\vdots$

$P_n \rightarrow$ divargance.

Proof:-

$$|P_1 - P| = |g(P_0) - g(P)| = \underbrace{|g'(c)|}_{<1} (P_0 - P) < (P_0 - P)$$

by mean value.

$\rightarrow P_1$ is closer to P from P_0.

P_1 اقرب لـ P من P_0

$$|P_n - P| = |g(P_{n-1}) - g(P)| = \underbrace{|g'(c)|}_{<k} (P_{n-1} - P) \le k (P_{n-1} - P) \le k \cdot k |P_{n-2} - P|$$

حسب النظرية

$\le k \cdot k \cdot k |P_{n-3} - P|$

$\vdots$

$\le k^n |P_0 - P|$.

$\rightarrow |P_n - P| \le k^n |P_0 - P|$

$\rightarrow \lim_{n \to \infty} |P_n - P| = 0$

$\rightarrow \lim_{n \to \infty} P_n = P$.

(2) $|P_1 - P| = \underbrace{|g'(c)|}_{>1} |P_0 - P| > |P_0 - P|$.

Theorem:-

k is the upper bound للمشتقة

a. $\underbrace{|P_n - P|}_{\text{error}} \le k^n |P_0 - P|$.

فنقدر نطلع upper bound for error

$\rightarrow$ we can found n

$\boxed{k = g'(P)} \rightarrow$ اذا منعرف P.

b. $|P_n - P| \le \dfrac{k^n |P_1 - P_0|}{1 - k}$. (excersise).

example:-

$x^3 - x + 5 = 0$.

Use Fixed point itteration to find all the roots, Find k for each case

$F(x) = x^3 - x + 5$

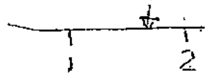
$F(0) = 5$

$F(-1) = 5$

$F(-2) = -11$

$F(2) = 1$

فوجد root



$x^3 = x + 5$

$x = \sqrt[3]{x+5} = (x+5)^{1/3}$

$g(x) = x$

$g'(x) = \frac{1}{3}(x+5)^{-2/3}$

$= \frac{1}{3\sqrt[3]{(x+5)^2}} < 1$ for all x, $0 < x$

$P_0 = 1.5$

فنوصل لل ata

for $x > 0$

$x + 5 > 5$

$(x+5)^2 > 25$

$\sqrt[3]{(x+5)} > (25)^{\frac{1}{3}} > 2$

$\frac{1}{\sqrt[3]{(x+5)}} < \frac{1}{2}$

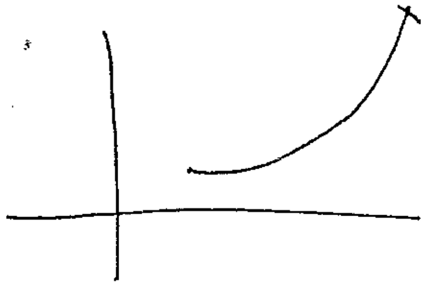
$\frac{1}{3\sqrt[3]{(x+5)}} < \frac{1}{6}$ $\quad K = \frac{1}{6}$

Discussion

- $f(x) = 1 + e^{-\cos(x-1)}$

$[1,2]$.

maxpoint. مهم كيف نطلع



5.

$x^4 - 3x^2 - 3 = 0$

10^{-2}

$P_0 = 1$

$[1,2]$.

$x^4 = 3x^2 + 3$.

$x = \sqrt[4]{3x^2 + 3}$.

$P_1 = g(1) = \sqrt[4]{6} = 1.56509$

$P_2 = 1.79358$

$P_3 = 1.88595$

$P_4 = 1.92285$

$P_5 = 1.93751$

$\cancel{P_6 = 1.94332}$

لزمنا 5 itteration حتى ثبتنا منزلتين

4 (12.2) a.

$$P_n = P_{n-1} - \frac{P_{n-1}^5 - 7}{5P_{n-1}^4}$$

$$g(x) = x - \frac{x^5 - 7}{5x^4}$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g(x) = x - \frac{x}{5} + \frac{7}{5x^4}$$

$$g(x) = \frac{4x}{5} + \frac{7}{5x^4}$$

$$g'(x) = \frac{4}{5} - \frac{28}{}$$

$P = 7^{1/5}$ $\quad P_n = g(P_{n-1})$

$x = 7^{1/5}$.

$x^5 = 7$

$x^5 - 7 = 0$.

$f(x) = x^5 - 7$

2.2, 2.4, ~~2.5~~ . افهم مادة
2.1.

$$g'(7^{1/5}) = \frac{4}{5} - \frac{28}{5(7^{1/5})^5}$$

$$= \frac{4}{5} - \frac{28}{5.7} = \frac{4}{5} - \frac{4}{5} = 0.$$ أسرع method
newton method.

14
2.2

Solve $x = \tan x$ in $[4,5]$.
$g(x) = \sec^2 x > 1$.

$x = \tan^{-1} x$

$g(x) = \tan^{-1} x$

$g'(x) = \frac{1}{1+x^2} < 1$

$P_0 = 4.5$.

$P_1 = \tan^{-1}(4.5)$

$= 1.352127$

$P_2 = \tan^{-1}(P_1)$

$= 0.93$

$x = \tan x = \tan(x-\pi) = \tan(x+\pi)$

$x = \tan(x-\pi)$

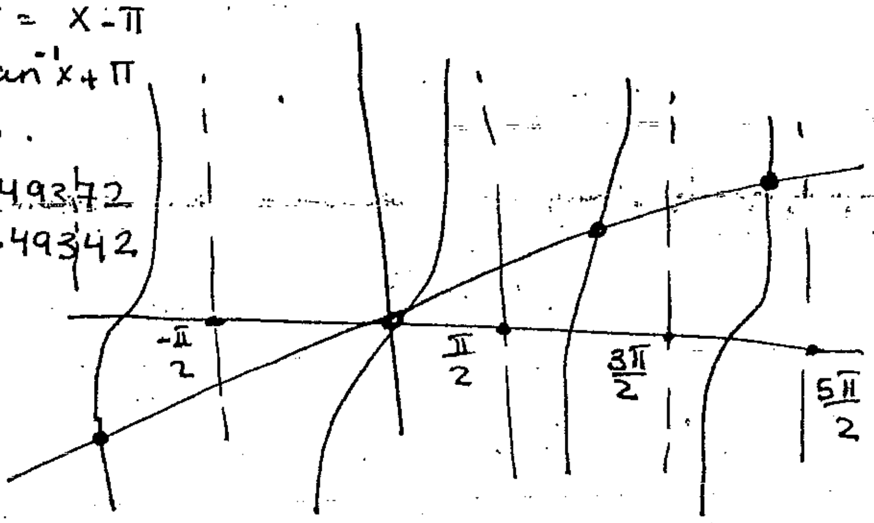
$\tan^{-1} x = x - \pi$

$x = \tan^{-1} x + \pi$

$P_0 = 4.5$.

$P_1 = 4.49372$

$P_2 = 4.49342$



14
2.1

Let $f(x) = (x-1)^{10}$
$P = 1$
$P_n = 1 + \frac{1}{n}$

Show that if $|f(P_n)| < 10^{-3}$ for $n > 1$
but $|P - P_n| < 10^{-3}$ requires $n > 1000$

1. $|P - P_n| < \epsilon$
2. $|c_{n+1} - c_n| < \epsilon$
3. $|f(P_n)| < \epsilon$

$F(p_n) = \left(\frac{1}{n}\right)^{10} < 10^{-3}$ For $n > 1$.

$|P - P_n| < 10^{-3} = |1 - 1 - \frac{1}{n}| < 10^{-3}$

$|-\frac{1}{n}| < 10^{-3}$

$\frac{1}{n} < 10^{-3} \Rightarrow n > 1000$.

$\frac{15}{2.1}$

$P_n = \sum_{k=1}^{\infty} \frac{1}{k}$

show that P_n divarge even though $\lim_{n\to\infty} (P_n - P_{n-1}) = 0$.

$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

$\lim_{n\to\infty} P_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$ harmonic series (divarge)

$P_n - P_{n-1} = (1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n}) - (1 + \frac{1}{2} + \cdots + \frac{1}{n-1}) = \frac{1}{n}$

$\lim_{n\to\infty} (P_n - P_{n-1}) = \lim_{n\to\infty} \frac{1}{n} = 0$.

$c_n = \frac{a_n + b_n}{2}$ stop.

$F(c_n) \leq \epsilon$ or $|c_n - c_{n-1}| \leq \epsilon$

stop if $F(c_n) \leq \epsilon$ and $\frac{c_n - c(n-1)}{c(n-1)} \leq 1 \times 10^{-6}$.

Solve this equ

$3x^2 - e^x = 0$

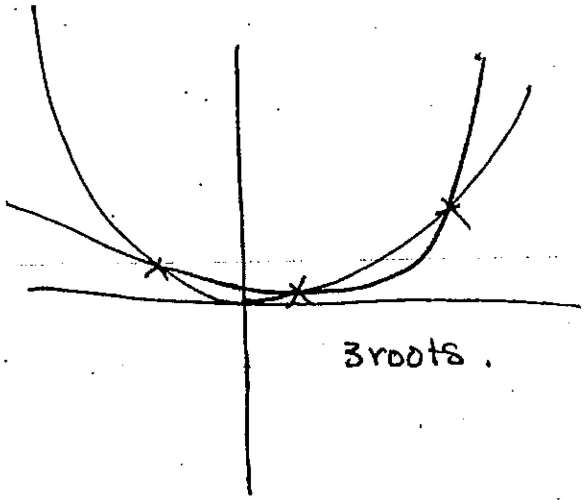
$f(0) = -1$

$f(1) = 0.28 > 0$

$f(2) = 12 - e^2 > 0$

$f(3) = 27 - e^3 > 0$

$f(4) = 48 - e^4 < 0$.



Newton method

$$f'(P_0) = \frac{f(P_0) - 0}{P_0 - P_1}$$

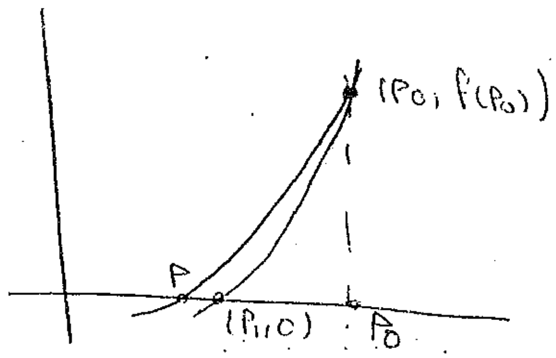
$$P_0 - P_1 = \frac{f(P_0)}{f'(P_0)}$$

$$P_1 = P_0 - \frac{f(P_0)}{f'(P_0)}$$

$$P_2 = P_1 - \frac{f(P_1)}{f'(P_1)}$$

$$\vdots$$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$



$$x = \underbrace{x - \frac{f(x)}{f'(x)}}_{g(x)} \leftarrow \text{Newton fixed point function}$$

Th:- Newton Raphson theorem

assume $f \in C^2[a,b]$ and $\exists\, P \in [a,b]$ such that $f(P)=0$, if $f'(P) \neq 0$ then there exist a $\delta > 0$ such that the sequance $\{P_k\}_{k=0}^{\infty}$ which is defined by $P_k = g(P_{k-1}) = P_{k-1} - \frac{f(P_{k-1})}{f'(P_{k-1})}$ will convarge to P for any initial approximation $P_0 \in [P-\delta, P+\delta]$

example:-

estimate $5^{\frac{3}{7}}$

$$x = 5^{3/7}$$

$$x^7 = 5^3$$

$$f(x) = x^7 - 125$$

$$P_1 = P_0 - \frac{f(P_0)}{f'(P_0)}$$

$f'(x) = 7x^6$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$$= P_n - \frac{P_n^7 - 125}{7P_n^6}$$

$$= \frac{6}{7} P_n + \frac{125}{7P_n^6}$$

$P_0 = 2$

$$P_1 = \frac{6}{7}(2) + \frac{125}{7(2)^6} = 1.71429$$

$$P_2 = \frac{6}{7}(1.71429) + \frac{125}{7(1.71429)^6} = 2.17$$

$\vdots$

Proof the theorem

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}$$

$$g'(x) = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$g'(P) = \frac{f(P)f''(P)}{(f'(P))^2} = 0$$

→ by Fixed point ~~itteration~~ theorey → the Fixed point itteration will convarge.

- if $e_{n+1} \approx A e_n$ where $e_n = P - P_n$ (error)

or $e_{n+1} \approx \frac{1}{100} e_n$ (error smaller than the first) (the best one)

$e_{n+1} \approx \frac{1}{2} e_n$ كلما صغر كلما اصبح احسن

كثير أص ← $e_{n+1} \approx A e_n^2$

if $e_n = 0.01$

$e_{n+1} = A(0.0001)$

Definition

P is a root of multiplicity M of $f(x)$ if $f(x) = (x-P)^M h(x)$, $h(P) \neq 0$.

- $f(x) = x^3 - 3x + 2$

  1 is a root of $f(x)$

  what is the multiplicity of 1?

$$
\begin{array}{r|l}
 & x^2 + x - 2 \\
\hline
x-1 & x^3 - 3x + 2 \\
 & -\ x^3 + x^2 \\
\hline
 & x^2 - 3x + 2 \\
 & -\ x^2 \pm x \\
\hline
 & -2x + 2 \\
 & +2x \mp 2 \\
\hline
 & 0
\end{array}
\qquad
\begin{array}{r|l}
 & x + 2 \\
\hline
x-1 & x^2 + x - 2 \\
 & -\ x^2 \pm x \\
\hline
 & 2x - 2 \\
 & 2x - 2 \\
\hline
 & 0
\end{array}
$$

$f(x) = (x-1)(x^2 + x - 2)$

1 has multiplicity 2 (quadratic root) $M=2$

-2 is a simple root $(M=1)$

Theory:-

P is a root of multiplicity M of $f(x)$ iff

$f(P)=0$, $f'(P)=0$, ... $f^{(m-1)}(P)=0$ but

$f^{(m)}(P) \neq 0$

Example:-

$f(x) = x^3 - 3x + 2$

$f(1) = 0$

$f'(x) = 3x^2 - 3$

$f'(1) = 0$

$f''(x) = 6x$

$f''(1) = 6$

$M = 2$

$$e_{n+1} \cong A e_n$$

$$\frac{e_{n+1}}{e_n} \approx A$$

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = A \rightarrow \text{linear Convergence.}$$

if $e_{n+1} \approx A e_n^2$

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^2} \approx A \rightarrow \text{quadratic Convergence.}$$

Definition:- Order of Convergence

assume $P_n \rightarrow P$ and $e_n = P - P_n$, if there exists two positive numbers A, R such that

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^R} = A$$

Then the sequance is said to converge to P with order of Convergence R, A is called the Asymptotic error Constant.

if $R=1$, we call it linear Convargance.

if $R=2$, we call it quadratic convargance.

example:-

Show that $P_n = \frac{1}{n^3}$ Convarges to $\overset{P}{0}$ linearly??

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{(n+1)^3}\right|}{\left|0 - \frac{1}{n^3}\right|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3}$$

$$= \left(\lim \frac{n}{n+1}\right)^3 = 1$$

$$\frac{1}{n^3} \xrightarrow[\text{convarge to}]{} 0 \text{ linearly}$$

Example:-

$f(x) = x^{101} - x^{100} - x + 1$

$f(1) - 0$

$f'(x) = 101x^{100} - 100x^{99} - 1$

$f'(1) = 101 - 100 - 1 = 0$.

$f''(x) = (101)(100)\, x^{99} - (100)(99)\, x^{98}$

$f''(1) \neq 0$

$M = 2$.

Theorem:- Convargence of newton method

if we use newton itiration,

1. if p is a simple root, then

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^2} = \left| \frac{f''(p)}{2f'(p)} \right|$$

$\left[\begin{array}{l} p \text{ is a simple root} \\ \text{Convargence is quadratic} \\ A = \left|\dfrac{f''(p)}{2f'(p)}\right| , R = 2 \end{array} \right]$

2. if p has moltiplicity $M > 1$, then

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|} = \frac{M-1}{M}$$

$\left[\begin{array}{l} \text{convargence is linear} \\ A = \dfrac{M-1}{M} , R = 1 \end{array} \right]$

example

$f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$.

$f(x) = (x-1)^2 (x+2)$

$f''(x) = 6x$

-2 is a simple roots

Convargance is fast $\quad R = 2 \left(\dfrac{|e_{n+1}|}{|e_n|^2} = \left| \dfrac{f''(p)}{2f'(p)} \right| \right)$

$$A = \left| \frac{f''(-2)}{2f'(-2)} \right| = \left| \frac{-12}{2(9)} \right| = \frac{2}{3}.$$

$P = 1$, $M = 2$

linear convargance ($R=1$)

$\lim \; |e_{n+1}| - 1$

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$P_0 = -2.4$

n	P_n	e_n ($P - P_0$)	$\frac{\lvert e_{n+1} \rvert}{\lvert e_n \rvert^2}$
0	-2.4	0.4	
1	-2.0761904	0.0761904	0.4761 ....
2	-2.00359601	0.003596	0.6194 ...
3	-2.00000858	0.000008589	0.6642 $\downarrow$ $\frac{2}{3} \approx A$

fast convargance.

$P_0 = 1.2$

n	P_n	e_n	$\frac{\lvert e_{n+1} \rvert}{\lvert e_n \rvert}$
0	1.2	-0.2	
1	1.103030	-0.10303	0.515 ...
2	1.0523564	-0.052356	0.5081
3	1.0264008	-0.02640811	0.4962 $\downarrow$ $A \approx \frac{1}{2}$.

slow convargance.

$A \longrightarrow \frac{1}{2}$.

Theory:- accelarated newton method

if p is a root of multiplicity M then the iteration

$$P_{n+1} = P_n - \frac{M f(P_n)}{f'(P_n)}$$ will convarge quadratically to p.

Ex:-

For the previous example. $f(x) = (x-1)^2 (x+2)$

1 has multiplicity 2, if we use the accelarated Newton itteration

$$P_{n+1} = P_n - \frac{2 f(P_n)}{f'(P_n)}$$ will get quadratic convargance!

$P_0 = 1.2$

n	P_n	e_n	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert^2}$
0	1.2	-0.2	
1	1.0060606	-0.00606	0.15
2	1.000006087	-0.000006087	0.15

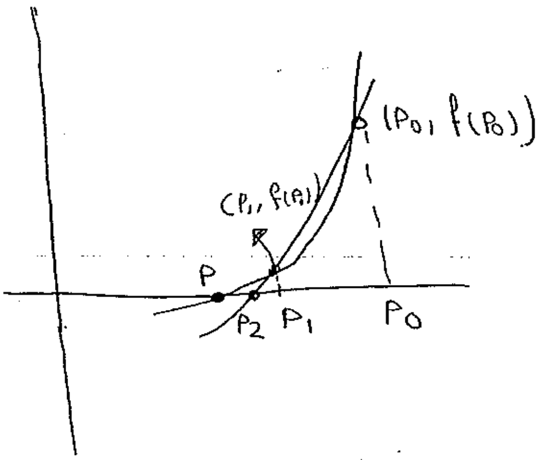
Proof $f(p) = 0$, $g'(p) = 0$, $g(p) = p$

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(c)}{2}(x-p)^2$$

$$P_{n+1} = g(P_n) = p + 0 + \frac{g''(c)}{2}(P_n - p)^2$$

Secant method:-

$$\frac{f(P_1)-0}{P_1-P_2} = \frac{f(P_1)-f(P_0)}{P_1-P_0}$$



$$P_1 - P_2 = \frac{f(P_1)(P_1-P_0)}{f(P_1)-f(P_0)}$$

$$P_2 = P_1 - \frac{f(P_1)(P_1-P_0)}{f(P_1)-f(P_0)}$$

$$P_3 = P_2 - \frac{f(P_2)(P_2-P_1)}{f(P_2)-f(P_1)}$$

$$\vdots$$

$$\boxed{P_n = P_{n-1} - \frac{f(P_{n-1})(P_{n-1}-P_{n-2})}{f(P_{n-1})-f(P_{n-2})}}$$

Theorem:-

if we use secant method to get $P_n \to P$ then.

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^{1.618}} = \left|\frac{f''(P)}{2f'(P)}\right|^{0.618}$$

$$\longrightarrow R = 1.618 = \frac{1+\sqrt{5}}{2}.$$

Ex:-

$f(x) = (x+2)(x-1)^2$.

$P_0 = -2.6$, $P_1 = -2.4$

and we use secant method.

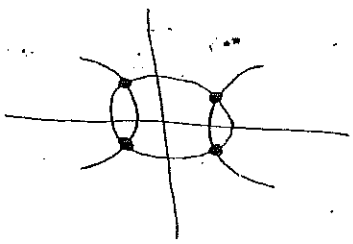
n	P_n	e_n	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert^{1.618}}$
0	-2.6	0.6	
1	-2.4	0.4	
2	-2.106598	0.106598	
3	-2.02264	0.02264	
4	-2.00151	0.00151	

	False position method	Secant method	Newton method
Speed	1	1.6	2
Coast	1	1	2
Convargance	very accurate	depends on P_0, P_1	depends on P_0.

2.6 Fixed point itteration For system of equation

$$x^2\cos y + y\sin x = 10$$
$$y\ln x + x^2\cos y = 5$$

$$x^2 - y^2 = 1$$
$$x^2 + y^2 = 2$$
$$2x^2 = 3.$$
$$x^2 = \frac{3}{2}.$$
$$x = \pm\sqrt{\frac{3}{2}}$$



$$x^2 - y^2 = x+3$$
$$x^2 + y^2 = e^x - 1.$$
$$2x^2 = x+3+e^x-1.$$
$$2x^2 - x - e^x - 2 = 0$$

$$x = g_1(x, y)$$
$$y = g_2(x, y)$$
$$(P_0, q_0)$$
$$P_1 = g_1(P_0, q_0) \qquad q_1 = g_2(P_0, q_0)$$
$$P_2 = g_1(P_1, q_1) \qquad q_2 = g_2(P_1, q_1).$$
$$P_{n+1} = g_1(P_n, q_n)$$
$$q_{n+1} = g_2(P_n, q_n)$$

Definition:-

(p, q) is a Fixed Point of the system
$x = g_1(x,y)$, $y = g_2(x,y)$ if $p = g_1(p,q)$ and $q = g_2(p,q)$

Def:-

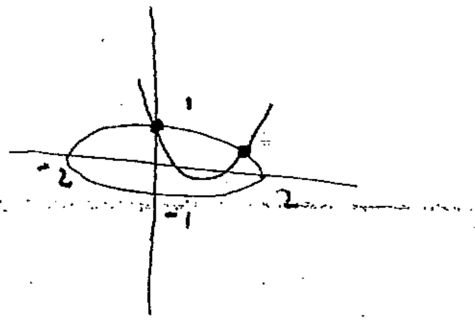
Fixed point itteration for the system
$x = g_1(x,y)$, $y = g_2(x,y)$ is given (p_0, q_0) then

$$p_{n+1} = g_1(p_n, q_n)$$
$$q_{n+1} = g_2(p_n, q_n) \qquad n = 1, 2, 3, \dots$$

Ex:-

$$f_1(x,y) = x^2 - 2x - y + 0.5 = 0$$
$$f_2(x,y) = x^2 + 4y^2 - 4 = 0 \quad \rightarrow x^2 + 4y^2 = 4$$
$$\frac{x^2}{4} + y^2 = 1.$$

estimate the solutions ?



$$x = \frac{x^2 - y + 0.5}{2} = g_1(x,y).$$

$$y = \frac{-x^2 - 4y^2 + 8y + 4}{8} = g_2(x,y).$$

$(p_0, q_0) = (0,1)$.

$p_1 = g_1(0,1) = \frac{0 - 1 + 0.5}{2} = -0.25$

$q_1 = g_2(0,1) = \frac{0 - 4 + 8 + 4}{8} = 1$.

$\vdots$

$p_4 = -0.2221680$ $\qquad$ $p_5 = -0.222194$

$q_4 = 0.9938121$ $\qquad$ $q_5 = 0.9938095$.

$(p_0, q_0) = (2, 0)$ (divarges).

$p_1 = g_1(2,0) = 2.25$

$q_1 = g_2(2,0) = 0$

$\vdots$

Let $g_1(x,y) = \frac{-x^2 + 4x + y - 0.5}{2}$

$g_2(x,y) = \frac{-x^2 - 4y^2 - 11x + 4}{11}$

$\downarrow$

$(p_0, q_0) = (2,1) \longrightarrow$ Convarge.

$(2,1) \longrightarrow (1.900, 0.311)$

Th:- Fixed point itteration. For system of equation:-

assume $g_1(x,y)$, $g_2(x,y)$ and their partial derivative are continous on a region that contains the Fixed point (P, q), if the starting point (P_0, q_0) is chosing sufficiently closed to (P, q) and.

$$\left|\frac{dg_1}{dx}\right| + \left|\frac{dg_1}{dy}\right| < 1 \text{ and } \left|\frac{dg_2}{dx}\right| + \left|\frac{dg_2}{dy}\right| < 1 \text{ in that region}$$

then the FPI will converge.

• Note:-

if (P, q) is given we apply the condition at (P, q) only.

Fixed point → we talk about g's

Newton → we talk about F.

to proof

المثال السابق

if $|x| < 0.5$ and $0.5 < y < 1.5$ أولاً نختار الفترة.

$$\left|\frac{dg_1}{dx}\right| + \left|\frac{dg_2}{dy}\right| = |x| + 0.5 < 1$$

أكبر قيمة ل x 0.5

$$\left|\frac{dg_2}{dx}\right| + \left|\frac{dg_2}{dy}\right| = \frac{|x|}{4} + |1-y| < \frac{1}{8} + 0.5 < 1$$

أكبر قيمة 0.5 ل x

أكبر قيمة 1.5 ل y

حتى نثبت ان النقطة المختارة divargance ← نختار فترة لا تحقق الشرطين السابقين او لا تحقق شرط واحد على الأقل.

example (linear system)

$$3x + 2y + 7z = 10 \rightarrow x = \frac{10 - 2y - 7z}{3} = g_1(x,y,z)$$

$$2x + 4y - z = 4 \rightarrow y = \frac{4 + z - 2x}{4} = g_2(x,y,z)$$

$$x + 5y + 10z = 15 \rightarrow z = \frac{15 - x - 5y}{10} = g_3(x,y,z).$$

$$P_1 = g_1(P_0, q_0, r_0)$$
$$q_1 = g_2(P_0, q_0, r_0)$$
$$r_1 = g_3(P_0, q_0, r_0).$$

$$\left.\begin{aligned} P_1 &= g_1(P_1, q_0, r_0) \\ q_1 &= g_2(P_1, q_0, r_0) \\ r_1 &= g_3(P_1, q_0, r_0) \end{aligned}\right\} \text{Gauss-sidel method}$$

- $P_{n+1} = P_n - \dfrac{f(P_n)}{f'(P_n)}$

$$f_1(x,y) = 0$$
$$f_2(x,y) = 0$$

$$\begin{pmatrix} P_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} P_n \\ q_n \end{pmatrix} - \underbrace{J^{-1}_{(P_n, q_n)}}_{\text{Jacubian}} \begin{pmatrix} f_1(P_n, q_n) \\ F_2(P_n, q_n) \end{pmatrix}$$

$$h : (x,y) \longrightarrow (f_1(x,y), f_2(x,y))$$

$$h' = J = \begin{pmatrix} \dfrac{dF_1}{dx} & \dfrac{dF_1}{dy} \\ \dfrac{dF_2}{dx} & \dfrac{dF_2}{dy} \end{pmatrix}$$

$$\boxed{\vec{\mathbb{P}}_{n+1} = \vec{\mathbb{P}}_n - J^{-1}\,\mathbb{F}}$$

2.7 Newton method

given $F_1(x,y)=0,\ F_2(x,y)=0$

and $F_1(P,q)=0,\ F_2(P,q)=0$

starting with (P_0,q_0) close to (P,q) then using taylor expansion in Two dimension at (P_0,q_0)

$$f_1(x,y) \cong f_1(P_0,q_0) + \left.\frac{df_1}{dx}\right|_{(P_0,q_0)}(x-P_0) + \left.\frac{dF_1}{dy}\right|_{(P_0,q_0)}(y-q_0)$$

$$F_2(x,y) \cong F_2(P_0,q_0) + \left.\frac{dF_2}{dx}\right|_{(P_0,q_0)}(x-P_0) + \left.\frac{dF_2}{dy}\right|_{(P_0,q_0)}(y-q_0)$$

subsitude (P,q) above

$$0 = F_1(P_0,q_0) + \left.\frac{df_1}{dx}\right|_{(P_0,q_0)}(P-P_0) + \left.\frac{df_1}{dy}\right|_{(P_0,q_0)}(q-q_0)$$

$$0 = F_2(P_0,q_0) + \left.\frac{dF_2}{dx}\right|_{(P_0,q_0)}(P-P_0) + \left.\frac{df_2}{dy}\right|_{(P_0,q_0)}(q-q_0)$$

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}f_1(P_0,q_0)\\ f_2(P_0,q_0)\end{bmatrix} + \underbrace{\begin{bmatrix}\frac{df_1}{dx} & \frac{df_1}{dy}\\ \frac{df_2}{dx} & \frac{df_2}{dy}\end{bmatrix}_{(P_0,q_0)}}_{J^{-1}}\begin{bmatrix}P-P_0\\ q-q_0\end{bmatrix}$$

$$\boxed{-\begin{bmatrix}F_1\\F_2\end{bmatrix}_{(P_0,q_0)} = J_{(P_0,q_0)}\begin{bmatrix}P-P_0\\ q-q_0\end{bmatrix}}$$ → Direct method.

$$-J^{-1}\begin{bmatrix}F_1\\F_2\end{bmatrix} = \begin{bmatrix}P-P_0\\ q-q_0\end{bmatrix}$$

$$\begin{bmatrix}P_0\\q_0\end{bmatrix} - J^{-1}_{(P_0,q_0)}\begin{bmatrix}F_1\\F_2\end{bmatrix}_{(P_0,q_0)} = \begin{bmatrix}P_1\\q_1\end{bmatrix}$$ inverse way.

- Inverse method.

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} - J^{-1}_{(p_n,q_n)} \begin{bmatrix} F_1(p_n, q_n) \\ F_2(p_n, q_n) \end{bmatrix}$$

- Direct method

$$-\begin{bmatrix} F_1(p_n, q_n) \\ F_2(p_n, q_n) \end{bmatrix} = J_{(p_n,q_n)} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$\Delta x = p_{n+1} - p_n \rightarrow p_{n+1} = \Delta x + p_n$$

$$\Delta y = q_{n+1} - q_n \rightarrow q_{n+1} = \Delta y + q_n$$

- <u>example</u>

Solve used Newton method.

- inverse method.

$$x^2 - 2x - y = 0.5 \rightarrow x^2 - 2x - y - 0.5 = 0 = f_1(x,y).$$

$$x^2 + 4y^2 = 4. \rightarrow x^2 + 4y^2 - 4 = 0 = f_2(x,y).$$

$$(p_0, q_0) = (2, 0.25)$$

$$J = \begin{pmatrix} 2x-2 & -1 \\ 2x & 8y \end{pmatrix}_{(2,0.25)} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}.$$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$$

$$f_1(2, 0.25) = 0.25$$

$$f_2(2, 0.25) = 0.25.$$

$$= \begin{pmatrix} 2 \\ 0.25 \end{pmatrix} - \frac{1}{8}\begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix}.$$

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1.90625 \\ 0.3125 \end{pmatrix} - \begin{pmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{pmatrix}^{-1} \begin{pmatrix} 0.008789 \\ 0.024414 \end{pmatrix}.$$

$$= \begin{pmatrix} 1.900691 \\ 0.311213 \end{pmatrix}$$

→ Direct method

$$-\begin{pmatrix} f_1(2,0.25) \\ f_2(2,0.25) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} Dx \\ Dy \end{pmatrix}$$

$$-\begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} Dx \\ Dy \end{pmatrix}.$$

$$DX = \frac{\begin{vmatrix} -0.25 & -1 \\ -0.25 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}} = \frac{-0.75}{8} = -0.09375$$

$$\begin{aligned} P_1 &= Dx + P_0 . \\ &= -0.09375 + 2 . \\ &= 1.90625 . \end{aligned}$$

$$Dy = \frac{\begin{array}{c|cc} \cancel{-0.25} & 2 & -0.25 \\ \cancel{-0.25} & 4 & -0.25 \end{array}}{8} = \frac{-0.5+1}{8} = \frac{0.5}{8} = +0.0625$$

$$Dy = q_1 + q_0$$

$$\begin{aligned} q_1 &= Dy + q_0 . \\ &= 0.0625 + 0.25 \\ &= 0.3125 . \end{aligned}$$

discussion

3.4

$\boxed{10}$ $f(x) = (x-p)^m h(x)$.

$\leftrightarrow$ $f(p)=0,\ f'(p)=0\ \cdots\ f^{(m-1)}(p)=0$ but $f^{(m)}(p) \neq 0$.

$f(p)=0$.

$f'(x) = m(x-p)^{m-1} h(x) + (x-p)^m h'(x)$.

$f'(p)=0$.

$f(p)=0$.

$(x-p)$ is a Factor of $f(p)$.

$(x-p)^2$ is a factor of $f'(p)$

$\boxed{8}$ $g(x) = x - \dfrac{m f(x)}{f'(x)}$. it will convarge quadrotically to p.

p is a root of multiplicity m for $f(x)$.

$g'(p)=0$ بدنا نثبت ان

$$f(x) = (x-p)^m h(x),\ h(p) \neq 0.$$

$$f'(x) = m(x-p)^{m-1} h(x) + (x-p)^m h'(x).$$

$$g(x) = x - \frac{m(x-p)^m h(x)}{m(x-p)^{m-1} h(x) + (x-p)^m h'(x)}$$

$$= x - \frac{m(x-p) h(x)}{m h(x) + (x-p) h'(x)}$$

$$g'(x) = 1 - \frac{(m h(x) + (x-p) h'(x))(m h(x) + (x-p) h'(x)) - m(x-p) h(x) \times \text{مشتقة المقام}}{[m h(x) + (x-p) h'(x)]^2}$$

$$g'(p) = 1 - \frac{(m h(p))^2}{m h(p)^2}$$

$$= 0.$$