Chapter 3

linear systems:-

Iteratine methods:-

1. Fixed point itteration
2. Gauss - sidel method
3. Newton Method.

- Direct methods:- (A is nonsingular).

1. Gaussian Elimenation $[A:b] \longrightarrow [U|c]$ + Back subsitution:
2. Gauss - Jordan $[A\backslash b] \longrightarrow [I\backslash x]$.
3. inverse method $x = A^{-1}b$
4. Gramer's :. $x_i = \frac{|A_i|}{|A|}$
5. L - U factorization.

Section 3.3

- back subsitution

$$3x_1 + 2x_2 + 4x_3 = 9$$
$$4x_2 + 6x_3 = 10.$$
$$10x_3 = 10.$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 0 & 4 & 6 & 10 \\ 0 & 0 & 10 & 10 \end{array}\right]$$

$10x_3 = 10 \longrightarrow \boxed{x_3 = 1}$

$4x_2 + 6x_3 = 10.$

$4x_2 = 10 - 6.$

$4x_2 = 4 \longrightarrow \boxed{x_2 = 1}$

$3x_1 + 2x_2 + 4x_3 = 9.$

$3x_1 = 9 - 4 - 2.$

$3x_1 = 3 \longrightarrow \boxed{x_1 = 1}$

$P_n = 10^{-2^n} \longrightarrow 0$ quodratically

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^n \cdot 2}} = 1$$

2. $P_n = 10^{-n^k} \not\longrightarrow 0$ quadratically.

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{(10^{-n^k})^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2(n^k)}}$$

$$= \lim_{n\to\infty} \frac{10^{2(n^k)}}{10^{(n+1)^k}} = \lim_{n\to\infty} \frac{10^{n^k} \cdot 10^{n^k}}{10^{(n+1)^k}} \longrightarrow \infty$$

$10 \leftarrow$ اسرع من البسط.

$\boxed{20}$
$\boxed{2.3}$

$$1,564,000 = 1,000,000\, e^{\lambda} + \frac{435,000}{\lambda}(e^{\lambda}-1)$$

$$1564 = 1000\, e^{\lambda} + \frac{435}{\lambda}(e^{\lambda}-1)$$

$$f(\lambda) = 1000\, e^{\lambda} + \frac{435}{\lambda}(e^{\lambda}-1) - 1564 = 0$$

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} & b_1 \\ \vdots & & & & & & \\ & & & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} & b_{n-2} \\ 0 & & & & a_{n-1,n-1} & a_{n-1,n} & b_{n-1} \\ & & & & & a_{n,n} & b_n \end{array}\right]$$

$a_{n,n}x_n = b_n$.

$$x_n = \frac{b_n}{a_{n,n}}.$$

$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$.

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}.$$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n-1}x_{n-1} - a_{n-2,n}x_n}{a_{n-2,n-2}}.$$

$$x_k = \frac{b_k - a_{k,k+1}x_{k+1} - a_{k,k+1}x_{k+2} - \cdots - a_{k,n}x_n}{a_{k,k}}$$

$$x_k = \frac{b_k - \sum_{j=k+1}^{n} a_{k,j}x_j}{a_{kk}} \quad k = n, 1.$$

$\boxed{3.3}$ Cost

Steps	+/-	×/÷
1	0	1
2	1	2
3	2	3
k	$k-1$	k
n	$n-1$	n
Total	$\frac{(n-1)(n)}{2}$	$\frac{n(n+1)}{2}$

$$\text{Total coast} = \frac{n^2-n}{2} + \frac{n^2+n}{2} = n^2;$$

3.4 Gaussian Elimination

$$AX = b$$

$$[A \backslash b] \longrightarrow [U \backslash c] + \text{back sub.}$$

Row operations:

1. multiply any row by a nonzero constant
2. switch any two rows
3. Replace any row by adding to it a nonzero multiple of another row

$$\text{row } r := \text{row } r + C \text{ row } p$$
$$= \text{row} - m_{r,p} \text{ row } p \quad ; \quad m_{r,p} = \frac{a_{r,p}}{a_{pp}} \quad r > p$$

Example

Solve:

$$x_1 + 2x_2 + x_3 + 4x_4 = 13$$
$$2x_1 + 4x_3 + 3x_4 = 28$$
$$4x_1 + 2x_2 + 2x_3 + x_4 = 20$$
$$-3x_1 + x_2 + 3x_3 + 2x_4 = 6.$$

Pivot element

Pivot Row

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right]$$

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$$
$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{4}{1} = 4$$
$$m_{41} = \frac{a_{41}}{a_{11}} = \frac{-3}{1} = -3$$

$$\begin{array}{r} \\ R_2 - 2R_1 \rightarrow \\ R_3 - 4R_1 \rightarrow \\ R_4 + 3R_1 \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & -6 & -2 & -15 & -32 \\ 0 & 7 & 6 & 14 & 45 \end{array}\right]$$

← pivot Row

$$m_{32} = \frac{a_{32}^{(2)}}{a_{22}^{(2)}} = \frac{6}{-4} = 1.5$$
$$m_{42} = \frac{a_{42}^{(2)}}{a_{22}^{(2)}} = \frac{7}{-4} = -\frac{7}{4} = -1.75$$

$$\begin{array}{r} \\ \\ R_3 - 1.5R_2 \rightarrow \\ R_4 + 1.75R_2 \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 9.5 & 5.25 & 48.5 \end{array}\right]$$

← pivot Row

$$m_{43} = \frac{a_{43}^{(3)}}{a_{33}^{(3)}} = \frac{9.5}{-5} = -1.9$$

$$\begin{array}{r} \\ \\ \\ R_4 - 1.9R_3 \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array}\right] \longrightarrow \begin{array}{l} x_4 = 2 \\ x_3 = 4 \\ x_2 = -1 \\ x_1 = 3 \end{array}$$

Coast

Step	$+/-$	$\times/\div$
1	4×3	$3+4\times3$
2	3×2	$2+3\times2$
3	2×1	$1+2\times1$
total	20	26

$\boxed{46}$

in general for $n\times n$ matrix

Step	$+/-$	$\times/\div$
1	$(n-1)n$	$(n-1)n+n-1$
2	$(n-2)(n-1)$	$(n-2)(n-1)+n-2$
3	$(n-3)(n-2)$	$(n-3)(n-2)+n-3$
⋮		
P	$(n-P)(n-P+1)$	$(n-P)(n-P+1)+n-P$
last step → $(n-1)$		

total $+/-$: $\sum_{P=1}^{n-1}(n-P)(n-P+1)$

$\times/\div$: $\sum_{P=1}^{n-1}(n-P)(n-P+1)+(n-P)$

$$\sum_{P=1}^{n-1}(n-P)(n-P+1)=\sum_{P=1}^{n-1}(n-P)^2+(n-P)$$

Let $k = n - p$

if $p = 1 \rightarrow k = n-1$

$p = n-1 \rightarrow k = 1$

$$\therefore \sum_{1}^{n-1} k^2 + k$$

$$\sum_{n=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6.}$$

$$\sum_{n=1}^{n} k = \frac{(n-1)n}{2.}$$

$$\therefore \sum_{1}^{n-1} k^2 + k = \frac{n(n+1)(2n+1)}{6} + \frac{(n-1)n}{2}$$

$$\text{total } \times / \div = \frac{(n-1)(n)(2n-1)}{6} + \frac{n(n-1)}{2} + \frac{n(n-1)}{2}$$

الكلي الاجمالي

$$\therefore \text{Grand Total} = 2\left[\frac{(n^2-n)(2n-1)}{6} + \frac{n(n-1)}{2}\right] + \frac{n(n-1)}{2}$$

$$= 2\left[\frac{2n^3-3n^2+n}{6} + \frac{3n^2-3n}{6}\right] + \frac{n^2-n}{2}$$

$$= \frac{2n^3-2n}{3} + \frac{n^2-n}{2}$$

$$= \frac{4n^3-4n+3n^2-3n}{6} = \frac{4n^3+3n^2-7n}{6}$$

$$\approx \frac{2}{3}n^3$$

- Coast for Gaussinan

$$\text{Coast} = \frac{4n^3+3n^2-7n}{6} + n^2$$

Coast For back Subsitution.

$$= \frac{4n^3+9n^2-7n}{6}$$

$$\approx \frac{2}{3}n^3$$

Algorithm

will store the Augumented matrix in $n+1$ coloumn.

$$\left[\begin{array}{cccc|c} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \cdots & a_{1,n}^{(1)} & a_{1,n-1}^{(1)} \\ a_{2,1}^{(1)} & a_{22}^{(1)} & \cdots & a_{2,n}^{(1)} & a_{2,n-1}^{(1)} \\ \vdots & & & \vdots & \vdots \\ a_{n,1}^{(1)} & a_{n,2}^{(1)} & \cdots & a_{nn}^{(1)} & a_{n,n-1}^{(1)} \end{array}\right]$$

and will construct an equivalent upper triangular. U

$$\left[\begin{array}{ccccc|c} a_{1,1}^{(1)} & a_{1,2}^{(1)} & & \cdots & a_{1,n}^{(1)} & a_{1,n-1}^{(1)} \\ 0 & a_{22}^{(2)} & & \cdots & a_{2,n}^{(2)} & a_{2,n-1}^{(2)} \\ 0 & 0 & a_{3,3}^{(3)} & \cdots & a_{3,n}^{(3)} & a_{3,n-1}^{(3)} \\ & 0 & & & a_{n,n}^{(n)} & a_{n,n-1}^{(n)} \end{array}\right]$$

<u>Step 1</u> Store the coefficient in array

<u>Step 2</u> switch rows if necessary so that $a_{1,1}^{(1)} \neq 0$

find $m_{r,1} = \dfrac{a_{r,1}^{(1)}}{a_{1,1}^{(1)}}$ for $r=2$ to n.

For c From 2 to $n+1$.

Set $a_{r,c}^{(2)} = a_{r,c}^{(1)} - m_{r,1} a_{1,c}^{(1)}$

we get

$$\left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1,n}^{(1)} & a_{1(n+1)}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2,n}^{(2)} & a_{2,n+1}^{(2)} \\ 0 & a_{32}^{(2)} & \cdots & a_{3,n}^{(2)} & a_{3,n+1}^{(2)} \\ \vdots & \vdots & & & \vdots \\ 0 & a_{n,2}^{(2)} & \cdots & a_{n,n}^{(2)} & a_{n,n+1}^{(2)} \end{array}\right]$$

<u>in general</u>

<u>P+1 step</u> find $a_{p,p}^{(p)} \neq 0$ From $r = p+1$ to N

$$m_{r,p} = \frac{a_{r,p}^{(p)}}{a_{p,p}^{(p)}} \text{ and } a_{r,p}^{(p+1)} = 0$$

For $c = p+1$ to $n+1$

$$a_{r,c}^{(p+1)} = a_{r,c}^{(p)} - m_{r,p} * a_{p,c}^{(p)}$$

we have 3 loop

Error

$0.37205 * (7) = 2.60435 \approx 2.6044$ نضرب بعدد صغير امن
$0.12345 * (7) = 0.86415 \approx 0.86415$ من الضرب بعدد كبير

- Gaussian elimination with pivoting:-

to avoid propagation of error we use the pivot element to be the largest in the remaining of the coloumn in $|a_{k,p}| = \max[|a_{pp}|, |a_{p+1,p}|$ ---- $|a_{n-1,p}|, |a_{n,p}|]$ and switch row p with row k if $k > p$

Example:-

$(1.000, 1.000)$ is a solution to
$$1.133x_1 + 5.281x_2 = 6.414$$
$$24.14x_1 - 1.210x_2 = 22.93$$

Solve the above by Gaussian with pivoting and without poviting.

- without pivoting

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & 1.210 & 22.93 \end{array}\right] \quad m_{21} = \frac{24.14}{1.133} = 21.31$$

$$\rightarrow \left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array}\right] \quad \begin{array}{l} x_2 = 1.001 \\ x_1 = 0.9956 \end{array}$$

with pivoting

$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 1.133 & 5.281 & \cancel{22.93}\ 6.414 \end{array}\right] \quad m_{21} = \frac{1.133}{24.14} = 0.0464$$

$$\rightarrow \left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array}\right] \quad \begin{array}{l} x_1 = 1.000 \\ x_2 = 1.000 \end{array}$$

$Ax = b$

- Gaussion $[A \backslash b] \longrightarrow [U \backslash c]$ + backsubsitution

2. Gauss - Jordan Elemination

$$\left[\begin{array}{ccc|c} \overset{1}{*} & *^{(1)} & *^{(1)} & *^{(1)} \\ \overset{0}{*} & *^{(1)} & *^{(1)} & *^{(1)} \\ \overset{0}{*} & *^{(1)} & *^{(1)} & *^{(1)} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & *^{(3)} & *^{(3)} \\ 0 & 1 & *^{(2)} & *^{(3)} \\ 0 & 0 & *^{(3)} & *^{(3)} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & *^{(4)} \\ 0 & 1 & 0 & *^{(4)} \\ 0 & 0 & 1 & *^{(4)} \end{array}\right]$$

Step	+/-	*/÷
1	3 × 2	3 + 3×2
2	2×2	2 + 2×2
3	1×2	1 + 1×2
⋮ n	~~[illegible]~~	

Solve

$$3x_1 + 2x_2 + 4x_3 = 9$$
$$x_1 - 2x_2 + 3x_3 = 2$$
$$3x_1 + 4x_2 - x_3 = 6$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 0 & -8/3 & 5/3 & -1 \\ 0 & 2 & -5 & -3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & 4/3 & 3 \\ 0 & 1 & -5/8 & 3/8 \\ 0 & 2 & -5 & -3 \end{array}\right]$$

Exercise

Find the total coast for Gauss Jord elimination

3. Inverse method.

$$[A \backslash I] \rightarrow [I \backslash A^{-1}]$$

$$\left[\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \middle| A^{-1}\right] \longrightarrow \left[\begin{matrix}*&*&*\\*&*&*\\*&*&*\end{matrix} \middle| \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right]$$

$Ax = b$

$x = A^{-1}b$

multiplication coast $= 2n^2 - n$.

Coast

Step	+/-	*/÷
1	5×2	$5+5\times2$
2	4×2	$4+4\times2$
3	3×2	$3+3\times2$
⋮	$(2n-p)(n-1)$	$(2n-p)(n-1)+(2n-p)$

$\longrightarrow$

Step	+/-	*/÷
1	$(2n-1)\times(n-1)$	$(2n-1)+(2n-1)(n-1)$
2	$(2n-2)(n-1)$	$(2n-2)+(2n-2)(n-1)$
p	$(2n-p)(n-1)$	$(2n-p)+(2n-p)(n-1)$
n	$n(n-1)$	$n+n(n-1)$

coast $= \dfrac{16n^3 - 9n^2 - n}{6} \approx 2\frac{2}{3}n^3$

4. Cramer's method

$$x_i = \frac{|A_i|}{|A|}$$

Find the coast of Cramer's method for 3*3 matrix.

$$x_1 = \frac{|A_1|}{|A|}$$

$$x_2 = \frac{|A_2|}{|A|}$$

$$x_3 = \frac{|A_3|}{|A|}$$

$4\,\underbrace{|\;|_{3\times3}}_{\text{determinant}} + \underbrace{3}_{\text{قسمة}}$

$$\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix} = a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

$4\times3+\underbrace{2}_{\text{جمع}} = 14$

Coast $= 4\times(14)+3$

$= 59$

3.6: L-U Factorization

$$Ax = b$$

$$LUX = b$$

1. $LY = b \longrightarrow$ Forward Subsitution
2. $UX = Y \longrightarrow$ backward subsitotion (coast $= n^2$).

$$[A] \longrightarrow \underbrace{\begin{bmatrix} a_{11}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} \cdots & a_{2n}^{(2)} \\ 0 & \cdots & a_{nn}^{(n)} \end{bmatrix}}_{U} \qquad L = \begin{bmatrix} 1 & & 0 \\ m_{12} & 1 & \\ m_{31} & m_{32} & 1 \\ \ddots & & & 1 \end{bmatrix}$$

Ex 3 -

Solve using L-U Factorization.

$$4x_1 + 3x_2 - x_3 = -2$$
$$-2x_1 + 4x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + 6x_3 = 7$$

no switch in Row.

$$\begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix} -2 \qquad m_{21} = -\frac{2}{4} = -\frac{1}{2}, \quad m_{31} = -\frac{1}{4} = 0.25$$

$$\xrightarrow[R_3 - 0.25R_1]{R_2 + 0.5R_1} \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 1.25 & 6.25 \end{bmatrix} \qquad m_{32} = \frac{1.25}{-2.5} = -0.5$$

$$\xrightarrow{R_3 + 0.5R_1} \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix} = U \qquad \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 0 \end{bmatrix} = L$$

$$LY = b.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 20 \\ 7 \end{bmatrix} \qquad \begin{aligned} y_1 &= -2 \\ y_2 &= 19 \\ y_3 &= 17 \end{aligned}$$

rward subsitution

$Ux = y$.

$$\begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.5 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 19 \\ 17 \end{bmatrix}$$

$x_3 = \frac{17}{8.5} = 2$

$x_2 = -4$

$x_1 = 3$

otal Coast =

w rations (Hiplication) $\frac{n^3 - n}{3}$ + backsubsitution n^2 + $\frac{n^2 - n}{}$ Forward subsitution + Row operation (addition) $* \frac{2n^3 - 3n^2 + n}{6}$

$$= \frac{2n^3 - 2n}{6} + \frac{6n^2}{6} + \frac{6n^2 - 6n}{6} + \frac{2n^3 - 3n^2 + n}{6}$$

$$= \frac{4n^3 + \cancel{12}^{9}n^2 - 7n}{6}.$$

step	+/-	*/÷
1	(n-1)(n-1)	(n-1) + (n-1)(n-1)
2	(n-2)(n-2)	(n-2) + (n-2)(n-2)
p	$(n-p)^2$	$(n-p) + (n-p)^2$

total $= \sum (n-p)^2 + \sum (n-p) + (n-p)^2.$

$= 2\sum (n-p)^2 + \sum (n-p)$

$= \frac{2(n-1)(n)(2n-1)}{6} + \frac{(n-1)(n)}{2}$

$= \frac{2(2n^3 - 3n^2 + n)}{6} + \frac{n^2 - n}{2}$

$= \frac{4n^3 - 6n^2 + 2n + 3n^2 - 3n}{6}$

$= \frac{4n^3 - 3n^2 - n}{6}$

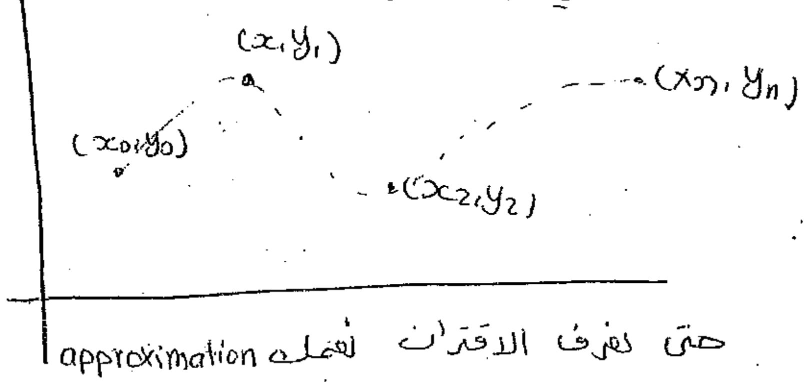
Chapter 4

Interpolation by polynomials:-

given $(x_0, y_0)\ (x_1, y_1)\ (x_2, y_2)\ \cdots\ (x_n, y_n)$

x_i	y_i
x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_n	y_n

ماهي العلاقة (الاقتران) بين هذه النقاط



حتى نعرف الاقتران نعمله approximation

Curve fitting أحسن خط يمر بين النقط بحيث تكون المسافة بينه النقط اصغر مايمكن

interpolation :- is estimation of the unknown Function by poly which passes through all given points

$$\boxed{P_n(x_i) = f(x_i)}$$

P_n is the approximation polynomial
f is the unknown Function

n من الدرجة Polonomial ← $(n+1)$ اذا عندنا points

Example $(1,2), (3,5), (7,10)$

3 نقاط ← الاقتران تربيعي

$P_2(x) = Ax^2 + Bx + C$

$P_2(1) = A + B + C = 2$

$P_2(3) = 9A + 3B + C = 5$

$P_2(7) = 49A + 7B + C = 10$

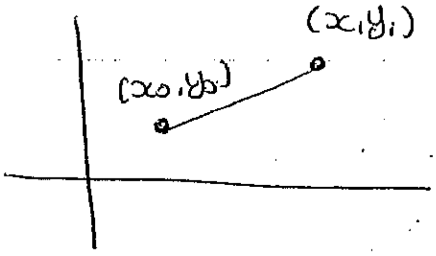
7

• given (x_0, y_0) (x_1, y_1) ... (x_n, y_n).

we need to Find the polynomial $P_n(x)$ which satisfies

$$\boxed{P_n(x_i) = y_i \quad , \quad i = 0, \dots, n}$$

• given (x_0, y_0), (x_1, y_1).



$$\text{Slope} = \frac{y_1 - y_0}{x_1 - x_0} = m$$

$$y - y_0 = m(x - x_0)$$

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \rightarrow y = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$= \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$$\cancel{P_1(x) = \frac{x - (x_1 + x_2)}{x_0 - (x_1 + x_2)} y_0 + \dots}$$

$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$P_3(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$P_n(x) = \underbrace{\frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)}}_{L_{n,0}} y_0 + \underbrace{\frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}}_{L_{n,1}} y_1$$

$$\dots + \underbrace{\frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}}_{L_{n,k}} y_k$$

$L_{n,k}$ → Lagrange coeficient polynomial

$$P_1(x) = \sum_{k=0}^{1} L_{1,k}(x)\, y_k$$

$$P_n(x) = \sum_{k=0}^{n} L_{n,k}(x)\, y_k \quad \leftarrow \text{Lagrange Polynomial}$$

Proof $\rightarrow$ $\boxed{P_n(x_i) = y_i}$

$n=1 \rightarrow \quad P_1(x_0) \overset{??}{=} y_0$

$$P_1(x_0) = y_0$$
$$P_1(x_1) = y_1$$

$n=2 \rightarrow$

$$P_2(x_0) = y_0$$
$$P_2(x_1) = y_1$$
$$P_2(x_2) = y_2$$

$n=k \rightarrow \boxed{P_n(x_k) = y_k}$

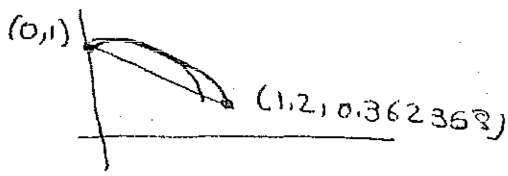
Example:-

given $f(x) = \cos x$ on $[0, 1.2]$

Find $P_1(x)$, $P_2(x)$, $P_3(x)$ and Compare the answers $P_1(0.35)$, $P_2(0.35)$, $P_3(0.35)$ to the exact.

to Find $P_1(x)$.

$$(x_0, y_0) = (0, \cos 0) = (0, 1)$$
$$(x_1, y_1) = (1.2, \cos 1.2) = (1.2, 0.362358)$$



$$P_1(x) = \frac{x - x_1}{x_0 - x_1}\, y_0 + \frac{x - x_0}{x_1 - x_0}\, y_1$$

$$= \frac{x - 1.2}{0 - 1.2}(1) + \frac{x - 0}{1.2 - 0}(0.362358)$$

$$P_1(x) = -0.833333(x - 1.2) + 0.301965\, x$$

$$P_1(0.35) = -0.833333(0.35 - 1.2) + 0.301965(0.35)$$
$$= 0.8140208$$

exact $= \cos(0.35) = 0.9393727$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$(0, \cos 0)$, $\cancel{(0.6, \cos 0.6)}$ $(0.6, \cos 0.6)$, $(1.2, \cos(1.2))$

$(0, 1)$, $(0.6, 0.825336)$, $(1.2, 0.362258)$

$$P_2(x) = \frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)}(1) + \frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)}(0.825336) + \frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)}(0.362258)$$

$$= 1.38889(x-0.6)(x-1.2) - 2.292599\,x(x-1.2) + 0.503275\,x(x-0.6)$$

$P_2(0.35) = 0.9233150528$

$h = \frac{1.2-0}{3} = 0.4$

$(0, 1)$, $(0.4, 0.921061)$, $(0.8, 0.696707)$, $(1.2, 0.362258)$

$$P_3(x) = \frac{(x-0.4)(x-0.8)(x-1.2)}{(0-0.4)(0-0.8)(0-1.2)}(1) + \frac{(x-0)(x-0.8)(x-1.2)}{(0.4-0)(0.4-0.8)(0.4-1.2)}(0.921061)$$

$$+ \frac{(x-0)(x-0.4)(x-1.2)}{(0.8-0)(0.8-0.4)(0.8-1.2)}(0.696707) + \frac{(x-0)(x-0.4)(x-0.8)}{(1.2-0)(1.2-0.4)(1.2-0.8)}(0.362258)$$

$$= -2.60417(x-0.4)(x-0.8)(x-1.2) \ldots$$

$$= 0.939607167$$

4:3 Langrange interpolating polynomial.

given $(x_0, y_0)\ (x_1, y_1)\ ----,\ (x_n, y_n)$

$$P_n(x) = \sum_{k=0}^{n} L_{n,k}(x) \cdot y_k = L_{n,0}(x)\, y_0 + L_{n,1}(x)\, y_1 + \cdots + L_{n,n}(x)\, y_n$$

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)(x-x_2) ---- (x-x_{k-1})(x-x_{k+1}) \cdots (x-x_n)}{(x_k-x_0)(x_k-x_1)(x_k-x_2) ---- (x_k-x_{k-1})(x_k-x_{k+1}) \cdots (x_k-x_n)}$$

Theory:-

if $f(x) = P_n(x) + E_n(x)$

$$E_n(x) = \frac{(x-x_0)(x-x_1) --- (x-x_n)}{(n+1)!} f^{(n+1)}(c)$$

for the previous example

$$E_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(c)$$

$(0,1), (0.6, 0.825336), (1.2, 0.362386)$

$$E_2(x) = \frac{x(x-0.6)(x-1.2)}{6} f^{(3)}(c).$$

$$E_2(0.35) = \frac{(0.35)(0.35-0.6)(0.35-1.2)}{6} f^{(3)}(c)$$

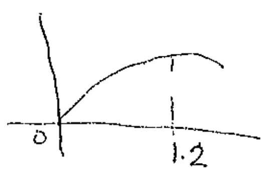
$$|E_2(x)| \leq \left| \frac{x(x-0.6)(x-1.2)}{6} \right| \max_{x_0 \leq x \leq x_n} |f^{3}(c)|$$

$f(x) = \cos x \quad [0, 1.2]$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x.$



$$\max_{0 \leq x \leq 1.2} |f'''(x)| = \sin(1.2) = 0.9320$$

$$|E_2(x)| \leq \frac{|x(x-0.6)(x-1.2)|}{6}(0.9320).$$

$$|E_2(0.35)| \leq \frac{0.35(0.25)(0.85)}{6}(0.9320)$$

$$= 0.01155$$

$P_2(0.35) = 0.93315$.

Exact $= 0.9393727$.

Error $= 0.0062$

Find an upperbound for $E_2(x)$ for all x.

فنشتق ونعوض المشتقة بالصفر ← نوجد x_{max}

ونعوض ← max upperbound.

$g'(x) = 0. \rightarrow$ uniform bound.

Theorem:- Uniform bound

For uniform partition $h = \frac{b-a}{n} = \frac{x_n - x_0}{n}$ n: integer

$\rightarrow x_k = x_0 + kh$

$x_n = x_0 + nh$

Let $M_n = \max\limits_{a \leq x \leq b} |f^{(n)}(x)|$

then 1. $|E_1(x)| \leq \frac{h^2 M_2}{8}$ for all $x \in [x_0, x_1]$

2. $|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$ for all $x \in [x_0, x_2]$

3. $|E_3(x)| \leq \frac{h^4 M_4}{24}$ for all $x \in [x_0, x_3]$

Ex:-

Using the theorem for the previous example.

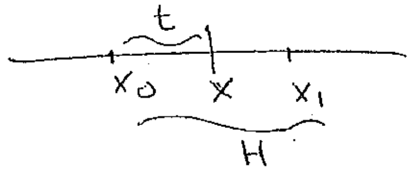
$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} = \frac{(0.6)^3(0.9320)}{9\sqrt{3}} = 0.03587$$

$\downarrow$

Upper bounded for all x

Proof $\quad |E_1(x)| \leq \frac{h^2 M_2}{8}$ for all $x \in [x_0, x_1]$

to show $|E_1(x)| \leq \frac{h^2 M_2}{8}$



$$E_1(x) = \frac{(x-x_0)(x-x_1)\, M_2}{2!}$$

$$h(x) = (x-x_0)(x-x_1).$$

let $t = x - x_0$

$$H(x) = t(x-x_1)$$

$$= t(t-h)$$

$x - x_1$ *

$= (x_0 + t) - (x_0 + h)$

$$H(t) = t(t-h).$$

$$H(t) = t^2 - th$$

$$H'(t) = 2t - h = 0$$

$t = \frac{h}{2} \longrightarrow$ critical point

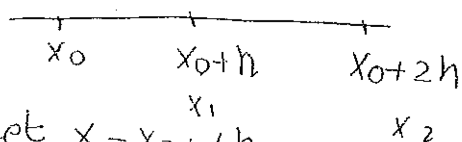
$$H(\overset{x}{\cancel{t}}) = \frac{h}{2}\left(\frac{h}{2} - h\right)$$

$$= \frac{h}{2}\left(-\frac{h}{2}\right) = -\frac{h^2}{4}$$

$$\max|H(x)| = \frac{h^2}{4}.$$

$$E_1(x) \leq \frac{h^2}{4} \cdot \frac{M_2}{2} = \frac{h^2 M_2}{8}$$

proof $\quad |E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}}$



let $x = x_0 + th$

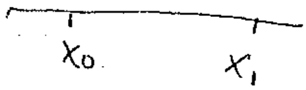
$0 \leq t \leq 2$

Theorem

$f(x) = P_n(x) + E_n(x)$.

then

$$E_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(c).$$

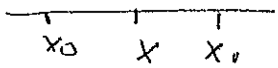


proof: for $n=1$.

To ~~should~~ show that the error.

$$E_n(x) = f(x) - P_n(x).$$

is equal to $\frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c)$.



Let $h(t) = f(t) - P_1(t) - E_1(x)\dfrac{(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)}$

$h(x)$ are continous and differentiable.

$h(x_0) = f(x_0) - P_1(x_0) - E_1(x)\dfrac{\cancel{(x_0-x_0)}(x_0-x_1)}{\cancel{(x-x_0)}(x-x_1)}$

$h(x_1) = f(x_1) - P_1(x_1) - 0 = 0$

$h(x) = f(x) - P_1(x) - E_1(x)\dfrac{(x-x_0)(x-x_1)}{(x-x_0)(x-x_1)}$

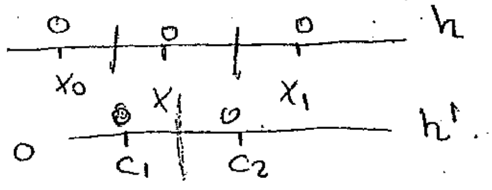
$= \underbrace{f(x) - P_1(x)}_{E_1(x)} - E_1(x)$

$h(x) = 0$.

$h(x) = 0$

Using MVT on (x_0, x), $\exists\, c \in (x_0, x)$

such that



$h'(c_1) = \dfrac{h(x) - h(x_0)}{x - x_0} = 0$

Similarly $\exists\, c_2 \in (x, x_1)$ such that

$$h'(c_2) = \frac{h(x_1) - h(x)}{x_1 - x} = 0$$

Similarly $\exists\, c \in (c_1, c_2)$ such that

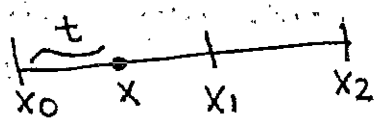
$$h''(c) = \frac{h'(c_2) - h'(c_1)}{c_2 - c_1} = \frac{0-0}{c_2-c_1} = 0.$$

• proof $\quad |E_2(x)| \leqslant \dfrac{h^3 m_3}{9\sqrt{3}}$

$x = x_0 + t$

$x_1 = x_0 + h$

$x_2 = x_0 + 2h$



$$|E_2(x)| = \frac{(x-x_0)(x-x_1)(x-x_2)\, f^{3}(c)}{3!}$$

$$|E_2(x)| \leqslant \frac{(x-x_0)(x-x_1)(x-x_2)\, M_3}{6}$$

$x - x_0 = t$

$x - x_1 = t - h$

$x - x_2 = t - 2h$

$$|E_2(x)| \leqslant \frac{(t)(t-h)(t-2h)\, M_3}{6}$$

$\phi(t) = t(t-h)(t-2h)$

$= (t^2 - th)(t-2h)$

$= t^3 - 2ht^2 - ht^2 + 2h^2 t = t^3 - 3ht^2 + 2h^2 t$

$\phi'(t) = 3t^2 - 6ht + 2h^2$

$\phi'(t) = 0$

$$t = \frac{6 \pm \sqrt{36 - 4\times 3\times 2}}{6}\, h$$

$t = 0.42264973\, h$

$t = 1.57735026\, 9 h$

$\phi(t) = 0.384900179\, h^3$

$$|E_2(x)| \leqslant \frac{0.384900179\, h^3 M_3}{6}$$

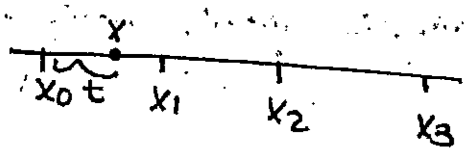
$|E_2(x)| \leqslant h^3\, M_3$

• proof $|E_3(x)| \leq \dfrac{h^4 M_4}{24}$

$$E_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\, f^{4}(c)}{4!}$$

$$|E_3(x)| \leq \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\, M_4}{24}$$

$x = x_0 + t$
$x_1 = x_0 + h$
$x_2 = x_0 + 2h$
$x_3 = x_0 + 3h$



$x - x_0 = t$
$x - x_1 = t - h$
$x - x_2 = t - 2h$
$x - x_3 = t - 3h$

$$|E_3(x)| \leq \frac{(t)(t-h)(t-2h)(t-3h)\, M_4}{24}$$

Let $\phi(t) = t\,(t-h)(t-2h)(t-3h)$
$= (t^2 - th)(t-2h)(t-3h)$
$= (t^3 - 2t^2h - t^2h + 2th^2)(t-3h)$
$= (t^3 - 3t^2h + 2th^2)(t-3h)$
$= t^4 - 6ht^3 + 11h^2t^2 - 6h^3t$

$\phi'(t) = 4t^3 - 18ht^2 + 22h^2t - 6h^3$

$\phi'(t) = 0$

$t = 2.618033989h$
$= 0.381966011h$
$= 0.05h$

$\phi(t) = |-1| = 1$

For $\phi(t)$ the max $= h^4$ @ $t = 2.618033989h$

$$|E_3(x)| \leq \frac{h^4 M_4}{24}$$

$$h'(t) = f'(t) - P_1'(t) - E_1(x)\left(\frac{(t-x_0)+(t-x_1)}{(x_0-x_1)(x-x_1)}\right)$$

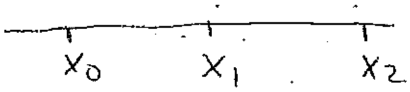
$$h''(t) = f''(t) - 0 - \frac{E_1(x)(2)}{(x_0-x_1)(x-x_1)}$$

$P_1''(t)$ because the function is linear

$$\rightarrow h''(c) = f''(c) - \frac{2E_1(x)}{(x-x_0)(x-x_1)} = 0$$

$$E_1(x) = \frac{(x-x_0)(x-x_1)}{2} f''(c).$$

<u>for $n=2$</u>



<u>exercise</u>

$$h(t) = f(t) - P_2(t) - E_2(x)\frac{(t-x_0)(t-x_1)(t-x_2)}{(x-x_0)(x-x_1)(x-x_2)}.$$

$$h(x_0) = f(x_0) - P_2(x_0) - 0$$

$$h(x_1) = f(x_1) - P_2(x_1) - 0.$$

$$h(x_2) = f(x_2) - P_2(x_2) - 0.$$

$$h(x) = f(x) - P_2(x) - E_2(x).(1).$$

$$= 0$$

4.4 Newton interpolation polynomial

given $x_0, x_1, x_2, \ldots, x_n$.

$(x_0, y_0), (x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$.

$$P_n(x_i) = f(x_i)$$

$$P_1(x) = a_0 + a_1(x - x_0)$$

$$P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1).$$

$$P_3(x) = P_2(x) + a_3(x - x_0)(x - x_1)(x - x_2).$$

$$\vdots$$

$$P_n(x) = P_{n-1}(x) + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}).$$

$$P_1(x_0) = a_0 + a_1(x_0 - x_0).$$

$$P_1(x_0) = a_0 = f(x_0) = y_0 \rightarrow \boxed{a_0 = y_0}$$

$$P_1(x_1) = \underset{a_0}{\underset{\parallel}{f(x_0)}} + a_1(x_1 - x_0) = f(x_1).$$

$$\boxed{a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}} = F[x_0, x_1]$$ First divided differance.

$$f(x_2) = P_2(x_2) = f(x_0) + F[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1).$$

$$a_2 = \frac{f(x_2) - f(x_0) - f[x_0, x_1)(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}.$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_0)} = F[x_0, x_1, x_2].$$

Then

$$a_k = f[x_0, x_1, x_2 \ldots x_k]$$ is k^{th} divided difference.

Definition

$F[x_k] = f(x_k)$ zeroth divided difference.

$$F[x_{k-1}, x_k] = \frac{f[x_k] - f[x_{k-1}]}{x_k - x_{k-1}} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$ 1st divided differance.

2nd divided difference.

$$F[x_{k-2}, x_{k-1}, x_k] = \frac{F[x_{k-1}, x_k] - f[x_{k-2}, x_k]}{x_k - x_{k-2}}$$

3rd divided difference

$$F[x_{k-3}, x_{k-2}, x_{k-1}, x_k] = \frac{F[x_{k-2}, x_{k-1}, x_k] - F[x_{k-3}, x_{k-2}, \ldots]}{x_k - x_{k-3}}$$

example:

(1,3), (2,5), (4,7), (8,11), (9,15)

$$f[2,4,8] = \frac{f[4,8] - f[2,4]}{8-2} = \frac{\frac{f(8)-f(4)}{8-4} - \frac{f(4)-f(2)}{4-2}}{6}$$

$$= \frac{\frac{11-7}{4} - \frac{7-5}{2}}{6} = 0$$

$$f[1,2,4,8] = \frac{f[2,4,8] - f[1,2,4)}{8-1}$$

$$= \frac{\frac{f(4,8) - f(2,4)}{8-2} - \left[\frac{f(2,4) - f(1,2)}{4-1}\right]}{8-1}$$

another way

x_k	$f(x_k)$ $f[x_k]$	1'st divided $f[x_{k-1}, x_k]$	2nd divided $f[x_{k-2}, x_{k-1}, x_k]$	3rd divided $F[x_{k-3}, x_{k-2}, x_{k-1}, x_k]$
x_0	$f(x_0)$ a_0			
x_1	$f(x_1)$	$f[x_0, x_1]$ a_1		
x_2	$f(x_2)$	$f[x_1, x_2]$	$f[x_0, x_1 x_2]$ a_2	
x_3	$f(x_3)$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	a_3 $F[x_0, x_1, x_2, x_3$
x_4	$f(x_4)$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$F[x_1, x_2, x_3, x_4$
x_5	$f(x_5)$	$f[x_4, x_5]$	$f[x_3, x_4, x_5]$	$F[x_2, x_3, x_4, x_5$
x_6	$f(x_6)$	$f[x_5, x_6]$	$f[x_4, x_5, x_6]$	$F[x_3, x_4, x_5, x_6$
⋮		⋮		

example

Find Newton interp $P_1, P_2, P_3, P_4, \ldots$ for the following table

x_k	$f(x_k)$	1'st	2'nd	3rd.	4th
1	-3 a_0				
2	0	3 a_1			
3	15	15	6 a_2		
4	48	33	9	1 a_3	
5	105	57	12	1	0
6	192	87	15	1	0

$P_1(x) = a_0 + a_1(x - x_0)$
$= -3 + 3(x-1)$

$P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1)$
$= -3 + 3(x-1) + 6(x-x_0)(x-2)$

$P_3(x) = P_2(x) + 1(x-1)(x-2)(x-$

$P_4 = P_3$

$P_5 = P_4 = P_3$

note:-

Error for newton interpolation polynomial equal to the Error for laggrage because they uses the same Polynomial.

Example:-

Estimate $f(5.5)$ using Newton int polynomial P_1, P_2, P_3, for the Following table.

x_k	$f(x_k)$	1st	2nd	3rd	4th
1	3				
3	4.5	0.75			
4.25	6		0.138462		
5.75	7.25			0.05722	
6	8				0.10287

$$P_1(x) = a_0 + a_1(x-x_0)$$
$$= 3 + 0.75(x-1).$$

$$f(5.5) \approx P_1(5.5)$$
$$= 3 + 0.75(5.5-1)$$
$$= 6.375.$$

$$P_2(x) = P_1(x) + a_2(x-1)(x-3)$$
$$= 0.13846(x-1)(x-3) + P_1(x)$$

$$f_2(5.5) \approx P_2(5.5)$$
$$\approx 6.375 + 0.13846(5.5-1)(5.5$$
$$\approx 7.93267$$

$$P_3(x) = P_2(x) + 0.05722(x-1)(x-3)(x-4.25)$$

$$f_3(5.5) \approx P_3(5.5)$$
$$\approx 7.93267 + 0.05722(5.5-1)(5.5-3)(5.5-4.25)$$
$$\approx 8.7373$$

$$P_4(x) = P_3 + 0.10287\,(x-1)(x-3)(x-4.25)(x-5.75)$$

$$\begin{aligned} f(5.5) &\approx P_4(5.5) \\ &\approx 8.7373 + 0.10287\,(4.5)(2.5)(1.25)(-0.25) \\ &\approx 8.375. \end{aligned}$$