Chapter Five

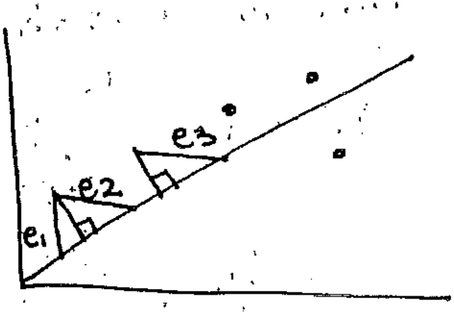
5.1+5.2

Best Fit

given $(x_0,y_0), (x_1,y_1) \dots (x_n,y_n)$ to Find the best Fitting cur

i.e

the curve with smallest distance to the given point



if $e_k = f(x_k) - y_k$

max error $E_\infty(F) = \|f\|_\infty = \max\limits_{0 \le k \le n} |e_k|$

Avarge error $= E_1 |f| = \|f\|_1 = (\sum |e_k|)/n$

Root Mean Square error $= E_2(f) = \|f\|_2 = (\sum |e_k|^2 / n)^{1/2}$

Example 5.1

Compare the max error, Avarge error and RMS error for the linear approximation $f(x) = -1.6x + 8.6$ to the data $(-1,10)$ $(0,9)$ $(1,7)$ $(2,5)$ $(3,4)$ $(4,3)$ $(5,0)$ $(6,-1)$

x_k	y_k	$f(x_k)$	$\|e_k\|$	e_k^2
-1	10	10.2	0.2	0.04
0	9	8.6	0.4	0.16
1	7	7	0	0
2	5	5.4	0.4	0.16
3	4	3.8	0.2	0.04
4	3	2.2	0.8	0.64
5	0	0.6	0.6	0.36
6	-1	-1	0	0

$\sum e_k = 2.6$

$E_\infty(F) = 0.8$

$E_1(F) = \dfrac{\sum |e_k|}{n}$

$= \dfrac{2.6}{8}$

$= 0.325$

$\sum e_k^2 = 1.4$

$E_2(f) = \left(\dfrac{\sum e_k^2}{n}\right)^{1/2}$

$= 0.42$

- to Find the best Fitting curve we need to minimize the least square error (RMS)

$$E_2(f) = \left(\frac{\sum_{k=1}^{n} |f(x_k) - y_k|^2}{n} \right)^{1/2}$$

$$n\, E_2^2(f) = \sum_{k=1}^{n} (f(x_k) - y_k)^2$$

$$E(?) = \sum_{k=1}^{n} (f(x_k) - y_k)^2$$

1. To Find The best Fitting line $f(x) = Ax+B$

$$E(A,B) = \sum_{k=1}^{n} |(Ax_k + B) - y_k|^2$$

$$\frac{dE}{dA} = \sum_{k=1}^{n} 2|(Ax_k + B) - y_k| \cdot x_k = 0 \quad --- (1)$$

$$\frac{dE}{dB} = \sum_{k=1}^{n} 2|(Ax_k + B) - y_k| \cdot 1 = 0 \quad --- (2)$$

(1) مِن $A \sum_{k=1}^{n} x_k^2 + B \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k x_k$ ---

(2) مِن $A \sum_{k=1}^{n} x_k + nB = \sum_{k=1}^{n} y_k$ ---

$$\left.\begin{array}{l} A \sum_{k=1}^{n} x_k^2 + B \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k x_k \\ A \sum_{k=1}^{n} x_k + nB = \sum_{k=1}^{n} y_k \end{array}\right\} \dashrightarrow \text{Normal equations}$$

Example:-

Find the best Fitting line $F(x) = Ax+B$ For the data (-1,10) (0,9) (1,7) (2,5) (3,4) (4,3) (5,0) (6,-1).

	x_k	y_k	x_k^2	$x_k y_k$
	-1	10	1	-10
	0	9	0	0
	1	7	1	7
	2	5	4	10
	3	4	9	12
	4	3	16	12
	5	0	25	0
	6	-1	36	-6
Σ	20	37	92	25

$$92A + 20B = 25$$
$$20A + 8B = 37$$

$$A = \frac{\begin{vmatrix} 25 & 20 \\ 37 & 8 \end{vmatrix}}{\begin{vmatrix} 92 & 20 \\ 20 & 8 \end{vmatrix}} \cong -1.61$$

$$B = \frac{\begin{vmatrix} 92 & 25 \\ 20 & 37 \end{vmatrix}}{\begin{vmatrix} 92 & 20 \\ 20 & 8 \end{vmatrix}} \cong 8.64$$

Example

for the following Data Find the best curve of the Form

$$y = Ax2$$

$$E(A) = \sum_{k=1}^{n} (Ax_k^2 - y_k)^2$$

$$\frac{dE}{dA} = 2\sum_{k=1}^{n} (Ax_k^2 - y_k) \cdot x_k^2 = 0$$

$$A = \frac{\sum_{k=1}^{n} y_k x_k^2}{\sum_{k=1}^{n} x_k^4}$$

التكملة :
مثل :
السابقا :

$$A = \frac{85}{2276} = 0.037346$$

Example

Find the best Fitting parrabola $f(x)= Ax^2+Bx+C$

$$E(A,B;C)=\sum_{k=1}^{n}\left[(Ax_k^2+Bx_k+c)-y_k\right]^2$$

$$\frac{dE}{dA}=0=2\sum_{k=1}^{n}\left[(Ax_k^2-Bx_k+c)-y_k\right]\cdot x_k^2$$

$$\frac{dE}{dB}=0=2\sum_{k=1}^{n}\left[(Ax_k^2+Bx_k+c)-y_k\right]\cdot x_k$$

$$\frac{dE}{dc}=0=2\sum_{k=1}^{2}\left[(Ax_k^2+Bx_k+c)-y_k\right]\cdot 1$$

5.2

linearization

$$f(x) \longrightarrow Ax+B$$

Example:-

Find the best Fitting curve of the form $f(x) = Ce^{Dx}$ for the following table. (0,1.5), (1,2.5), (2,3.5), (3,5), (4,7.5

$y = Ce^{Dx}$

$\ln y = \ln C + Dx$

$\ln y = Dx + \ln C$

$Y = AX + B$

$Y = \ln y$

$X = x$

$D = A$

$C = e^{B}$

x_k	y_k	X_k	$Y_k = \ln y_k$	X_k^2	$X_k y_k$
0	1.5	0	0.405465	0	0
1	2.5	1	0.916291	1	0.916291
2	3.5	2	1.25..	4	2.5
3	5	3	1.6	9	4.8281
4	7.5	4	2.01	16	8.059.
Σ		10	6.198860	30	16.30974

table 5.4 From the text book

$$30A + 10B = 16.309742$$
$$10A + 5B = 6.198860$$

$$A = \frac{\begin{vmatrix} 16.309742 & 10 \\ 6.198860 & 5 \end{vmatrix}}{\begin{vmatrix} 30 & 10 \\ 10 & 5 \end{vmatrix}} = 0.3912023$$

$$B = 0.457367.$$

$$D = A \cong 0.39$$

$$C = e^{B} = e^{0.457367} \cong 1.58$$

$$f(x) = 1.58\, e^{0.39x} = Ce^{Dx}$$

Examples:-

1. $y = \frac{D}{x+c}$

$\frac{1}{y} = \frac{x}{D} + \frac{c}{D}$.

$Y = AX + B$

$Y = \frac{1}{y}$; $X = x$, $A = \frac{1}{D}$; $B = \frac{c}{D}$.

$\downarrow$ $D = \frac{1}{A}$ $\downarrow$ $c = BD$

$y = \frac{D}{x+c}$

$y = \frac{1}{x} D + \frac{D}{c}$

$A = D/x$

$A = x$ Y/x

$A = D/x$

2. $y = \frac{x}{A+Bx}$

$\frac{1}{y} = \frac{A}{x} + B$.

$Y = AX + B$.

$Y = \frac{1}{y}$

$X = \frac{1}{x}$.

$A = A$

$B = B$.

3. $y = cxe^{-Dx}$

$\frac{y}{x} = c\,e^{-Dx}$

$\ln\left(\frac{y}{x}\right) = \ln c - Dx$

$\ln\left(\frac{y}{x}\right) = -Dx + \ln c$

$Y = A\cancel{X} + B$.

$Y = \ln\left(\frac{y}{x}\right)$

$X = x$

$A = -D \rightarrow D = -A$

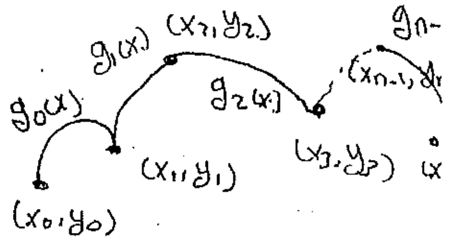
$B = \ln c \rightarrow c = e^{B}$.

Section 5.5

Cubic spline

given $(x_0, y_0), (x_1, y_1), \dots, (y_n, y_n)$

The cubic spline is a Function $g(x)$ such that it is a cubic polynomial between every two nodes and its of this Form $g_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$ on $[x_i, x_{i+1}]$ for $i = 0, 1, \dots, n-1$ and that satisfies



1. $g_i(x_i) = y_i \quad i = 0, 1, \dots, n-1, \; g_{n-1}(x_n) = y_n$
   $(n+1)$ conditions.

if I have n Function $\longrightarrow$ $4n$ unknowns.

2. $g_i(x_{i+1}) = g_{i+1}(x_{i+1}) \quad i = 0, \dots, n-2$
   $(n-1)$ conditions
   $g_0(x_1) = g_1(x_1)$
   $g_1(x_2) = g_2(x_2)$
   $g_{n-2}(x_{n-1}) = g_{n-2}(x_n)$.

-- continous
- نفس المشتقة
- نفس التقعر

3. $g_i'(x_{i+1}) = g_{i+1}'(x_i+1) \quad i = 0, \dots, n-2 \rightarrow (n-1)$ condition
4. $g_i''(x_{i+1}) = g_{i+1}''(x_{i+1}) \quad i = 0, \dots, n-2 \longrightarrow (n-1)$ condition.

so we have $(n+1) + (3(n-1)) = 4n-2$ conditions.

$\rightarrow$ ~~eq~~ Since $g_i(x_i) = y_i \Rightarrow$
$d_i = y_i$

$\longrightarrow$ equation (2) gives

$$y_{i+1} = g_{i+1}(x_{i+1}) = a_i(x_{i+1}-x_i)^3 + b_i(x_{i+1}-x_i)^2 + c_i(x_{i+1}-x_i) + d$$

✳ $$= a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i$$

where $h_i = (x_{i+1} - x_i)$

$g_i'(x) = 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i$

$$g''_i(x) = 6a_i(x-x_i) + 2b_i$$

if $\boxed{S_i = g_i''(x_i)}$

subsitute $\longrightarrow$ $\boxed{b_i = \frac{S_i}{2}}$

Using the same equation.

$$g''_i(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i$$

$$S_{i+1} = 6a_i h_i + S_i$$

$$\boxed{a_i = \frac{S_{i+1} - S_i}{6h_i}}$$

subsitote in $*$

$$c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6}$$

- Considering the equation

$$g'_i(x_i) = g'_{i+1}(x_i) \text{ we get}$$

$$h_{i-1} S_{i-1} + 2(h_{i-1} + h_i) s_i + h_i S_{i+1} = 6\left[f(x_i, x_{i+1}) - f[x_{i-1}, x_i] \right)$$

for $i = 1, \dots, n-1$

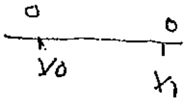
$$\begin{array}{c|cccccccc} & S_0 & S_1 & S_2 & S_3 & \cdots & S_{n-2} & S_{n-1} & S_n \\ & h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & h_1 & 2(h_1+h_2) & h_2 & \cdots & 0 & 0 & 0 \\ 3 & 0 & 0 & h_2 & 2(h_2+h_3) & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ & & & & & h_{n-3} & 2(h_{n-3}, h_{n-2}) & h_{n-2} & 0 \\ & & & & & & h_{(n-2)} & 2(h_{n-2}, h_{n-1}) & h_{n-1} \end{array} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \\ S_n \end{bmatrix} = 6 \begin{bmatrix} F[x_1, x_2] - F[x_0, x_1 \\ F[x_2, x_3] - F[x_1, x_2 \\ F[x_3, x_4] - F[x_2, \\ \vdots \\ F[x_n, x_{n+1}] - F[x_{n-1}, x \end{bmatrix}$$

$(n-1)$ equations × $(n+1)$ unknown, we need two more condition.

1. Natural Spline $S_0 = S_n = 0$

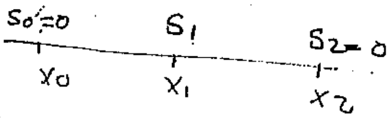
we get $(n-1)$ equations with $(n-1)$ Unknowns

When $n=1$ × ما بينفع لأنه مصفر على الجهتين



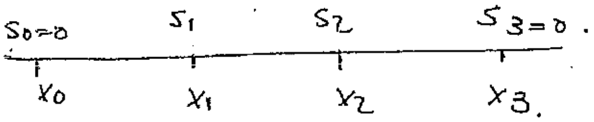
لا يوجد matrix لكن يوجد معادلة .

when $n=2$



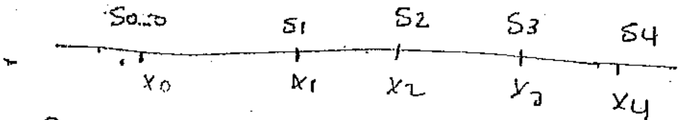
$$2\boxed{(h_0+h_1)}S_1 = 6\left[f[x_1,x_2] - f[x_0,x_1]\right).$$

when $n=3$



$$\begin{bmatrix} 2(h_0+h_1) & h_1 \\ h_1 & 2(h_1+h_2) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} f[x_1,x_2] - f[x_0,x_1] \\ f[x_2,x_3] - f[x_1,x_2] \end{bmatrix}$$

when $n=4$



$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 \\ h_1 & 2(h_1+h_2) & h_2 \\ 0 & h_2 & 2(h_1+h_3) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} f[x_1,x_2] - f[x_0,x_1]. \\ f[x_2,x_3] - f[x_1,x_2] \\ f[x_3,x_4] - f[x_2,x_3] \end{bmatrix}$$

Example

Find the natural Spline For the given table.

x_i	y_i
0	2
1	4.4366
1.5	6.7134
2.25	13.9130

$h_0 = 1, \quad h_1 = 0.5, \quad h_2 = 0.75$

$f[0,1] = 2.4366$

$F[1,1.5] = 4.5536$

$f[1.5, 2.5] = 9.5995$

$$\begin{bmatrix} 2(1.5) & 0.5 \\ 0.5 & 2(1.25) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} 4.5536 - 2.4366 \\ 9.5995 - 4.5536 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0.5 \\ 0.5 & 2.5 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 12.7020 \\ 30.2754 \end{bmatrix}$$

$S_0 = 0$
$S_3 = 0$.

$S_1 = \underline{2.292}, \quad S_2 = \underline{11.6618}$

$a_i = \frac{S_{i+1} - S_i}{6 h_i}$

$a_0 = \frac{S_1 - S_0}{6 h_0} = \frac{2.292 - 0}{6(1)} = 0.3820$

$a_1 = ??$

$a_2 = ??$

$b_i = \frac{S_i}{2}$

$b_0 = \frac{S_0}{2} = 0$

$b_1 = \frac{S_1}{2} = 1.146$

$b_2 = \frac{S_2}{2} = 5.8259$

$c_i = \dots$

$c_0 = 2.0546$

$c_1 = 3.2005$

$c_2 = 6.6866$

$d_i = y_i$

$d_0 = 2$

$d_1 = 4.4215$

$d_2 = 6.7130$

$g_0(x) = 0.3820(x-0)^3 + 0(x-0)^2 + 2.054(x-0) + 2.000$ on $[0,1]$

$g_1(x) = 3.1199(x-1)^3 + 1.146(x-1)^2 + 5.205(x-1) + 4.4366$ on $[1, 1.5$

$g_2(x) = -2.5895(x-1.5)^3 + 5.8259(x-1.5)^2 + 6.6866(x-1.5) + 6.7134$ on $[1.5, 2$

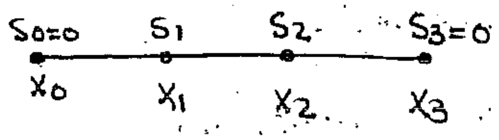
$f(0.66) = 3.4659$ Exact $= 3.34343$

$f(1.75) = 8.7087$ Exact $= 8.4467$

natural Spline

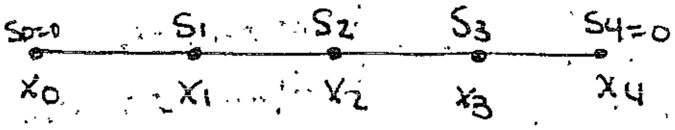
$S_0 = 0 \quad S_n = 0$

$\underline{n=3}$



$$\begin{bmatrix} 2(h_0+h_1) & h_1 \\ h_1 & 2(h_1+h_2) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} F(x_1,x_2) - F(x_0,x_1) \\ F(x_2,x_3) - F(x_1,x_2) \end{bmatrix}$$

$\underline{n=4}$



$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 \\ h_1 & 2(h_1+h_2) & h_2 \\ 0 & h_2 & 2(h_2+h_3) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} F(x_1,x_2) - F(x_0,x_1) \\ F(x_2,x_3) - F(x_1,x_2 \\ F(x_3,x_4) - F(x_2,x_3 \end{bmatrix}$$

• Clambed Spline

$F'(x_0)=A$

$F'(x_n)=B$

(1) $\longrightarrow$ $2h_0S_0+h_0S_1=6\left[F(x_0,x_1)-A\right]$

(2) $\longrightarrow$ $h_{n-1}S_{n-1}+2h_{n-1}S_n=6\left[B-F(x_{n-1},x_n)\right]$

$\underline{n=1}$

$$\begin{bmatrix}2h_0 & h_0\\ h_0 & 2h_0\end{bmatrix}\begin{bmatrix}S_0\\ S_1\end{bmatrix}=6\begin{bmatrix}f[x_0,x_1]-A\,\text{\textcircled{x_0}}\\ B-F[x_0,x_1]\end{bmatrix}$$

$g_0(x)=a_0(x-x_0)^3+b_0(x-x_0)^2+c_0(x-x_0)+b_0$

نعوض في النقاط وكذلك المشتقة عند الأطراف وبالتالي نعرف المجاهيل .

$\underline{n=2}$

$$\begin{bmatrix}2h_0 & h_0 & 0\\ h_0 & 2(h_0+h_1) & h_1\\ 0 & h_1 & 2h_1\end{bmatrix}\begin{bmatrix}S_0\\ S_1\\ S_2\end{bmatrix}=\text{\textcircled{6}}\begin{bmatrix}F(x_0,x_1)-A\\ F(x_1,x_2)-F(x_0,x_1)\\ B-\underline{F(x_1,x_2)}\end{bmatrix}$$

$$g(x)=\begin{cases}a_0(x-x_0)^3+b_0(x-x_0)^2+c_0(x-x_0)+d_0 & x_0\leq x\leq x_1\\ a_1(x-x_1)^3+b_1(x-x_1)^2+c_1(x-x_1)+d_1 & x_1\leq x\leq x_2\end{cases}$$

$f'(x_0)=A$

$f'(x_1)=B$

$f(x_0)=d_0=y_0$

$f(x_1)=d_1=y_1$

$g_0(x_1)=g_1(x_1)$

$g_0'(x_1)=g_1'(x_1)$

$g_0''(x_1)=g_1''(x_1)$

$g_1(x_2)=y_2$

$g_0(x_0)=y_0$

$g_1(x_1)=y_1$

$g_1(x_2)=y_2$

For $n=3$

$$\begin{bmatrix} 2h_0 & h_0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 \\ 0 & 0 & h_2 & 2h_2 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} F(x_0,x_1)-A \\ F(x_1,x_2)-F(x_0,x_1) \\ F(x_2,x_3)-F(x_1,x_2) \\ B-F(x_2,x_3) \end{bmatrix}$$

Q2) Clamped spline

$(0,0)$ $(1,1)$ $(2,2)$

$S'(0)=1$, $S'(2)=1$

$$g(x)=\begin{cases} g_0(x)=a_0(x-0)^3+b_0(x-0)^2+c_0(x-0)+d_0 & \text{on } [0,1] \\ g_1(x)=a_1(x-1)^3+b_1(x-1)^2+c_1(x-1)+d_1 & \text{on } [1,2] \end{cases}$$

$$g(x)=\begin{cases} g_0(x)=a_0x^3+b_0x^2+c_0x+d_0 & \text{on } [0,2] \\ g_1(x)=a_1(x-1)^3+b_1(x-2)^2+c_1(x-1)+d_1 & \text{on } [1,2] \end{cases}$$

$g_0[0]=d_0=0$

$g_1(2)=d_1=1$

$g_0'(x)=3a_0x^2+2b_0x+c_0$

$g_0'(0)=c_0=1$

$g_1'(x)=3a_1(x-1)^2+2b_1(x-1)+c_1$

$=3a_1+2b_1+c_1=2$

$g_0'(1)=g_1'(1)$

$3a_0x^2+2b_0x+c_0=3a_1(x-1)^2+2b_1(x-1)+c_1$

$3a_0+2b_0+1=3a_1(0)+2b_1(0)+c_1$

$g_0''(1) = g_1''(1)$

$6a_0x + 2b_0 = 6a_1(x-1)^2 + 2b_1$

$6a_0 + 2b_0 = 2b_1$

$g_1(2) = 2$

$a_1 + b_1 + c_1 = 2-1$

$a_1 + b_1 + c_1 = 1$

$g_0'(1) = g_1(1)$

$a_0 + b_0 + c_0 + d_0 = d_1$

$a_0 + b_0 + c_0 = 1$

0	0	//	///
1	1	1	///
1	2	1	0

$h_0 = 1$

$h_1 = 1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} = 6 \begin{bmatrix} 1-1 \\ 1-1 \\ 1-1 \end{bmatrix}$$

$S_0 = 0$

$S_1 = 0$

$S_2 = 0$.

$\boxed{Q15}$ cubic-poly $[a, b]$

$\overset{+}{a}$ $\overset{+}{b}$

$f(x)$ its own clamped spline but it cannot be its own Free spline ?

$f(a) =$

cubic $S_0, S_1 \neq$ zero

المشتقة الثانية ≠ صفر its not natural

المشتقة الأولى ≠ صفر

$a_3 \neq 0$

$g(x) = a_3x^3 + bx^2 + C_2x + d_2$

$g'(x) = 3a_3x^2 + 2bx + C_2$

$g''(x) = 6a_3x + 2b_2$

$= a_3 | b_2$ صفر عند الطرفين

ولكن $a_3 = 0$

4 - Unknowns

4 equ (condition)

$y_0 = a_0(x - x_0)^3 + b_0(x - x_0)^2 + c_0(x - x_0) + d_0$

أربع نقاط مشتركة وبالتالي poly نفسها