Chapter 9

Numerical Solution of 1^{st} order ODE's

1'st order ODE

- $y'(t) = f(t, y(t))$

  $y(t_0) = y_0$

  $y' = \frac{t-y}{2}$

  $y(0) = 1$

- $t^2 y' + \sin t \, y^2 = \cos t$

$$y(t_1) = y(t_0 + h)$$
$$= y(t_0) + h y'(t_0) + \frac{h^2}{2!} y''(c)$$

$\rightarrow y(t_1) \approx y(t_0) + h y'(t_0)$ with error $= \frac{h^2}{2!} y''(c)$

$$= y_0 + h f(t_0, y_0) \quad \text{(section 9.2)}$$

$$y(t_1) \approx y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(c) \quad \text{(Section 9.4)}$$

9.2 Euler method

Consider $y' = f(t, y)$

$y(t_0) = y_0$

we will approximate the solution using set of points (t_k, y_k) where

$$\underbrace{y_k}_{\text{Estimate}} \approx \underbrace{y(t_k)}_{\text{estimation at } t_k}$$

- we will use n subintervales of $[a, b]$

  $h = \frac{b-a}{n}$, $t_k = a + hk$, $k = 1, \ldots, n$

- Using taylor expansion of $y(t_1)$ at t_0,

$$y(t_1) = y(t_0+h) = y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(c)$$

$\rightarrow$ $y_1 = y_0 + h f(t_0, y_0)$ with step error $= \frac{h^2}{2} y''(c)$

notice that $y_1 \approx y(t_1)$

$$y_2 = y_1 + h f(t_1, y_1)$$
$$y_3 = y_2 + h f(t_2, y_2)$$

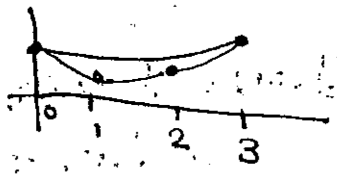
$$\vdots$$

$t_1 = t_0 + h$, $t_2 = t_0 + 2h$

$y_{n+1} = y_n + h f(t_n, y_n)$ Euler method, step error $= \frac{h^2}{2} y''(c)$

EXAMPLE

Estimate the solution of $y' = \frac{t-y}{2}$, $y(0) = 1$, on $[0, 3]$



$$y_1 = y_0 + h f(t_0, y_0)$$
$$= 1 + 1\, f(0,1) = 1 + (-0.5) = 0.5$$

$$y_2 = y_1 + h f(t_1, y_1)$$
$$= 0.5 + 1\, f(1, 0.5) = 0.5 + \frac{1-0.5}{2} = 0.75$$

$$y_3 = y_2 + h f(t_2, y_2)$$
$$= 0.75 + 1\, f(2, 0.75) = 0.75 + \frac{2-0.75}{2} = 1.375$$

Total error $= E(y(h), h)$

$$= \frac{y''(c_1) h^2}{2} + \frac{y''(c_2) h^2}{2} + \cdots + \frac{y''(c_n) h^2}{2}$$
$$= \frac{h^2}{2} \left(y''(c_1) + y''(c_2) + \cdots + y''(c_n) \right)$$
$$= \frac{h^2}{2} \left(n y''(c) \right)$$
$$= \frac{h^2}{2} \left(\frac{b-a}{h} y''(c) \right) = \frac{(b-a) h\, y''(c)}{2} \approx ch$$

$E\left(y(h), \frac{h}{2}\right) = c\left(\frac{h}{2}\right) = \frac{1}{2} ch = \frac{1}{2} E(y(h), h)$ نصف الخطأ السابق

9.4 Taylor method

Derive a formula of total error $O(h^2)$ to solve

$$y' = \frac{t-y}{2} \quad \text{on } [0,3]$$

$$y_0 = 1, \quad h = 1$$

$$y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} f''(t_0, y_0)$$

$$y(t_1) = \underbrace{y_0 + hy'(t_0) + \frac{h^2}{2} y''(t_0)}_{y_1} + \underbrace{\frac{h^3}{3!} y'''(c)}_{\text{Error}}$$

$$y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} \frac{df(t_0, y_0)}{dt}$$

Step error $= \frac{h^3}{6} y'''(c)$

$$y_{k+1} = y_k + hf(t_k, y_k) + \frac{h^2}{2} y''(t_k)$$

Total error $= E(y(h), h) = ch^2 = \frac{y'''(c)(b-a)h^2}{6}$

- Solving the example

$$y_1 = y_0 + hf(t_0, y_0) + \frac{h^2}{2} y''(t_0)$$

$$= 1 + 1f(0,1) + \frac{1}{2} y''(0)$$

$$= 1 + \frac{0-1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{(0-1)}{4}\right)$$

$$= 1 - 0.5 + 0.25 + 1/8 = 0.875$$

$$y'(t) = \frac{t-y}{2}$$

$$y''(t) = \frac{1}{2} - \frac{y'}{2}$$

$$= \frac{1}{2} - \frac{(t-y)}{4}$$

$$y_2 = y_1 + hf(t, y_1) + \frac{h^2}{2} y''(t_1)$$

$$= y_1 + hF(t, y_1) + \frac{h^2}{2} \frac{d}{dt}(f(1, 0.875))$$

$$= 0.875 + hf(1, 0.875) + \frac{1}{2}\left(\frac{1}{2} - \frac{1-0.875}{4}\right)$$

$$= 0.875 + 0.0625 + 0.25 - 0.03125$$

$$= 1.15625$$

$y'(t) = f(t, y(t))$

$y(t_0) = y_0$

Use taylor method of ordor 4 to estimate the solution of $y' = \frac{t-y}{2}$, $y(0)=1$ on $[0,3]$, $h=1$

$$y(t_1) = \underbrace{y(t_0) + hy'(t_0) + \frac{h^2}{2!} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y''''(t_0)}_{y_1} + \underbrace{\frac{h^5}{5!} y^{(5)}(c)}_{\text{steperror}}$$

- total error $= E(y(\cancel{t_0}), \cancel{t_0} h) = ch^4$

$$y_{k+1} = y_k + hy'(t_k) + \frac{h^2}{2!} y''(t_k) + \frac{h^3}{3!} y'''(t_k) + \frac{h^4}{4!} y^{(4)}(t_k)$$

$$y_1 = y_0 + hy'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{3!} y'''(t_0) + \frac{h^4}{4!} y''''(t_0)$$

$y'(t) = \frac{t-y}{2}$, $y'(0) = \frac{0-1}{2} = \frac{-1}{2}$ $(y_0 = 1)$

$y''(t) = \frac{1}{2}(1 - y') = \frac{1}{2}\left(1 - \left(\frac{t-y}{2}\right)\right) = \frac{1}{2} - \frac{t-y}{4}$

$y''(0) = \frac{1}{2} - \left(-\frac{1}{4}\right) = 0.75$

$y'''(t) = \frac{1}{2}(-y'') = -\frac{1}{2}\left(\frac{1}{2} - \frac{t-y}{4}\right)$

$y'''(0) = -\frac{1}{2}(0.75) = -0.375$

$y''''(t) = -\frac{1}{2} y''' = -\frac{1}{2}\left(-\frac{1}{2}\left(\frac{1}{2} - \frac{t-y}{4}\right)\right)$

$y''''(0) = -\frac{1}{2}(-0.375) = 0.1875$

$y_1 = 1 + 1(0.5) + \frac{1}{2}(0.75) + \frac{1}{6}(-0.375) + \frac{1}{24}(0.1875)$

$= 0.8203125$

$y_2 = y_1 + hy'(t_1) + \frac{h^2}{2} y''(t_1) + \frac{h^3}{3!} y'''(t_1) + \frac{h^4}{4!} y^{(4)}(t_1)$

$t_1 = 1$, $y_1 = 0.8203125$

$y_2 = 1.1045\ldots$

$y_3 = 1.670\ldots$

$E(y(b),h) = ch^4$

$E(y(b),h/2) = c(h/2)^4 = ch^4/16$

$E(y(b),10^{-2}h) = c(10^{-2}h)^4 = c(10^{-8})h^4$

taylor → one evaluation

- Modified Euler Method :- (Huen's Method)

$y'(t) = f(t, y(t))$

$y(t_0) = y_0$

1. $\int_{t_0}^{t_1} y'(t)\,dt = \int_{t_0}^{t_1} f(t, y(t))\,dt$ → Using trapozoidal

$y(t_1) - y(t_0) = \frac{h}{2}(f(t_0,y_0) + f(t_1, y(t_1)))$ Error $= \frac{-h^3 y''(c)}{12}$

$y(t_1) \simeq y_0 + \frac{h}{2}(f(t_0,y_0) + f(t_1, \underbrace{y(t_1)}_{\text{ندخل قيمة معروفة}}))$

$y(t_1) = y_0 + \frac{h}{2}(f(t_0,y_0) + f(t_1, y_0 + hf(t_0,y_0)))$

- $y_{k+1} = y_k + \frac{h}{2}(f(t_k, y_k) + f(t_{k+1}, y_k + hf(t_k, y_k)))$

total error For Huen's Method $= ch^2$

- EXAMPLE

Solve Using Huen's Method with $h=1$

$y' = \frac{t-y}{2} = f(t,y)$

$y(0) = 1$

$f(0,1)$

$y_1 = y_0 + \frac{h}{2}(f(t_0,y_0) + f(t_1, y_0 + hf(t_0,y_0)))$

$= 1 + \frac{1}{2}(f(0,1) + f(1, 1 + (1)f(0,1)))$

$= 1 + \frac{1}{2}(-0.5 + f(1, 1-0.5)) = 1 + \frac{1}{2}\left(-0.5 + \frac{1-0.5}{2}\right) = 0.875$

$$y_2 = y_1 + \frac{h}{2}\left(f(t_1, y_1) + f(t_2, y_1 + hf(t_1, y_1))\right)$$

$$= 0.875 + \frac{1}{2}\left(f(1, 0.875) + f(2, 0.875 + (1)\, f(1, 0.875))\right)$$

$$= 0.875 + \frac{1}{2}\left(\left(\frac{1-0.875}{2}\right) + f(2, 0.875(1 - 0.875/2))\right)$$

$$= 1.171875$$

$$y_3 = 1.732422$$

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if we have a period of $[0, 0.5]$, $h = \frac{1}{4}$

0	0.25	0.5
t_0	t_1	t_2

$$y_1 = 1 + 1/2\,(-0.25) = 0.875$$

on $[0, 1]$, $h = \frac{1}{2}$

0	0.5	1
t_0	t_1	t_2

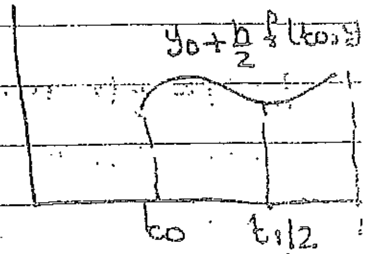
9.5 RK4: Range-Kutta Method of ordor 4

$$y'(t) = f(t, y(t))$$
$$y(t_0) = y_0$$

$$\int_{t_0}^{t_1} y'(t)\,dt = \int_{t_0}^{t_1} f(t, y(t))\,dt$$

$$y_{n+1} = y_n + \frac{h}{6}(f_0 + 2f_1 + 2f_2 + f_3) \quad \text{where}$$

$$f_0 = f(t_0, y_0)$$
$$f_1 = f(t_0 + h/2,\ y_0 + h/2\, f_0)$$
$$f_2 = f(t_0 + h/2,\ y_0 + h/2\, f_1)$$
$$f_3 = f(\cancel{t_0},\ y_0 + hf_2)$$



$$f_1 = \frac{f_1 + f_2}{2}$$

$$f_2(t_{1/2}, y_{1/2}) = y_0 + \frac{h}{2}$$
$$f_1(t_{1/2}, y_{1/2}) = y_0 + \frac{h}{2} f_1$$

EXAMPLE

Solve $y' = \frac{t - y}{2}$, $y(0) = 1$, $[0, 3]$, $h = \frac{1}{4}$

$$y_1 = y_0 + \frac{h}{6}(f_0 + 2f_1 + 2f_2 + f_3)$$

$$f_0 = f(t_0, y_0) = f(0, 1) = -0.5$$

$$f_1 = f(t_0 + h/2,\ y_0 + h/2\, f_0) = f\left(\tfrac{1}{8},\ 1 + 1/8(-0.5)\right) = -0.40625$$
$$f_2 = f(t_0 + h/2,\ y_0 + h/2\, f_1) = f(1/8,\ 1 + 1/8(-0.40625)) = -0.4$$
$$f_3 = f(t_1,\ y_0 + hf_2) = f(1/4,\ 1 + 1/4(-0.4121094)) = -0.3234863$$

$$y_1 = 1 + \frac{1/4}{6}\left((-0.5) + 2(-0.40625) + 2(-0.4121094) + (-0.3234863)\right)$$
$= 0.8974915$ comparing to the exact 0.8974917

$y_2 = y_1 + h/6 \, (f_0 + 2f_1 + 2f_2 + f_3)$..

$f_0 = f(t_1, y_1) = f(1/4, 0.8974915) = \ldots$

$f_1 = f(t_1 + h/2, \, y_1 + h/2 \, f_0) = f(3/8, \, 0.8974915 + 1/8 \, f_0) =$

$f_2 = f(\frac{3}{8}, \, 0.8974915 + \frac{1}{8} (f_1)) = \ldots$

$f_3 = f(1/2, \, 0.8974915 + 1/4 \, (f_2)) =$

$y_2 = \ldots$