$$Chapter 4$$

$$Interpolation:$$

$$Given \left(n+1\right) points:$$

$$\rightarrow \left(x\_{0},y\_{0}\right),\left(x\_{1},y\_{1}\right), … ,\left(x\_{n},y\_{n}\right)$$

$$x\_{0}<x\_{1}<x\_{2}<x\_{3}<…< x\_{n}$$

$$Result:$$

$$∃! P\_{n}\left(x\right) of degree at most \overline{n} ∴ P\_{n}\left(x\_{k}\right)=y\_{k} for k= 0, 1, 2, …$$

$$In this chapter two main interpolation methods :\left(Lagrange \& Newton\right)$$

1. $Lagrange interpolation:$

$$Case \#1 (2 points):$$

$$P\_{1}\left(x\right)=y\_{0}\left(\frac{x-x\_{1}}{x\_{0}-x\_{1}}\right)+y\_{1}\left(\frac{x-x\_{0}}{x\_{1}-x\_{0}}\right)$$

$$or we can write it as:$$

$$P\_{1}\left(x\right)=y\_{0}L\_{1,0}(x)+y\_{1}L\_{1,1}(x)$$

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$$Case \#2 (3 points):$$

$$P\_{2}\left(x\right)=y\_{0}\left(\frac{(x-x\_{1})(x-x\_{2})}{(x\_{0}-x\_{1})(x\_{0}-x\_{2})}\right)+y\_{1}\left(\frac{(x-x\_{0})(x-x\_{2})}{(x\_{1}-x\_{0})(x\_{1}-x\_{2})}\right)+y\_{2}\left(\frac{(x-x\_{0})(x-x\_{1})}{(x\_{2}-x\_{0})(x\_{2}-x\_{1})}\right)$$

$$or we can write it as:$$

$$P\_{2}\left(x\right)=y\_{0}L\_{2,0}\left(x\right)+y\_{1}L\_{2,1}\left(x\right)+y\_{2}L\_{2,2}(x)$$

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$$Following the same pattern:$$

$$\vdots $$

$$\vdots $$

$$\vdots $$

$$\vdots $$

$$Case \#n (n+1 points):$$

$$P\_{n}\left(x\right)=y\_{0}L\_{n,0}\left(x\right)+y\_{1}L\_{n,1}\left(x\right)+y\_{2}L\_{n,2}\left(x\right)+…+y\_{n}L\_{n,n}\left(x\right)$$

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1. $Newton^{'}s Interpolation:$

$$Before studing this interpolation we shall study the Divided Difference \left(D.D\right):$$

1. $Zeros D.D:$

$$f\left[x\_{k}\right]=y\_{k}=f\left(x\_{k}\right)$$

1. $First D.D:$

$$f\left[x\_{k-1},x\_{k}\right]=\frac{y\_{k}-y\_{k-1}}{x\_{k}-x\_{k-1}}$$

1. $Second D.D:$

$$f\left[x\_{k-2},x\_{k-1},x\_{k}\right]=\frac{f\left[x\_{k-1},x\_{k}\right]-f\left[x\_{k-2},x\_{k-1}\right]}{x\_{k}-x\_{k-2}}$$

$$\vdots $$

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$$Divided Difference Table:$$

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| $$x\_{k}$$ | $$0^{'}s D.D$$ | $$1^{st} D.D$$ | $$2^{nd} D.D$$ | $$3^{rd} D.D$$ | $$\cdots $$ | $$n^{th} D.D$$ |
| $$x\_{0}$$ | $$y\_{0}$$ | $$---$$ | $$---$$ | $$---$$ | $$\cdots $$ |  |
| $$x\_{1}$$ | $$y\_{1}$$ | $$F\left[1\right]=\frac{y\_{1}-y\_{0}}{x\_{1}-x\_{0}}$$ | $$---$$ | $$---$$ |  |
| $$x\_{2}$$ | $$y\_{2}$$ | $$F\left[2\right]=\frac{y\_{2}-y\_{1}}{x\_{2}-x\_{1}}$$ | $$\frac{F[2]-F[1]}{x\_{2}-x\_{0}}$$ | $$---$$ |  |
| $$x\_{3}$$ | $$y\_{3}$$ | $$F\left[3\right]=\frac{y\_{3}-y\_{2}}{x\_{3}-x\_{2}}$$ | $$\frac{F[3]-F[2]}{x\_{3}-x\_{1}}$$ | $$F[x\_{0},x\_{1},x\_{2},x\_{3}]$$ |  |
| $$\vdots $$ |
| $$x\_{n}$$ | $$y\_{n}$$ | $$F\left[n\right]=\frac{y\_{n}-y\_{n-1}}{x\_{n}-x\_{n-1}}$$ | $$\frac{F[n]-F[n-1]}{x\_{n}-x\_{n-2}}$$ |  | $$\cdots $$ |  |

$$Now back to the interpolation formula:$$

$$P\_{n}\left(x\right)=a\_{0}+a\_{1}\left(x-x\_{0}\right)+a\_{2}\left(x-x\_{0}\right)\left(x-x\_{1}\right)+…+a\_{n}\left(x-x\_{0}\right)\left(x-x\_{1}\right)…\left(x-x\_{n-1}\right)$$

$$∴ a\_{0}=F\left[x\_{0}\right]$$

$$ a\_{1}=F\left[x\_{0},x\_{1}\right]$$

$$ \vdots $$

$$ a\_{n}=F\left[x\_{0},x\_{1},…,x\_{n}\right]$$

$$It is easy to get the inerpolation coefficients \left(a\_{0}, …,a\_{n}\right) from the table : $$

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| $$x\_{k}$$ | $$0^{'}s D.D$$ | $$1^{st} D.D$$ | $$2^{nd} D.D$$ | $$3^{rd} D.D$$ | $$\cdots $$ | $$n^{th} D.D$$ |
| $$x\_{0}$$ | $$a\_{0}$$ | $$---$$ | $$---$$ | $$---$$ | $$\cdots $$ |  |
| $$x\_{1}$$ | … | $$a\_{1}$$ | $$---$$ | $$---$$ |  |
| $$x\_{2}$$ | … | … | $$a\_{2}$$ | $$---$$ |  |
| $$x\_{3}$$ | … | … | … | $$a\_{3}$$ |  |
| $$\vdots $$ |
| $$x\_{n}$$ | … | … | … | … | $$\cdots $$ | $$a\_{n}$$ |

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$$Interpolation Error:$$

$$In general the error can be obtained using this formula:$$

$$$$

$For the uniform partition we can use the following formulas:$

$1-when n=1:$

$$\left|E\_{1}(x)\right|\leq \frac{h^{2}M\_{2}}{8} ∴M\_{2}:the maximum value for |f^{\left(2\right)}(c)|$$

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$2-when n=2:$

$$\left|E\_{2}(x)\right|\leq \frac{h^{3}M\_{3}}{9\sqrt{3}} ∴M\_{3}:the maximum value for |f^{\left(3\right)}(c)|$$

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$3-when n=3:$

$$\left|E\_{3}(x)\right|\leq \frac{h^{4}M\_{4}}{24} ∴M\_{4}:the maximum value for |f^{\left(4\right)}(c)|$$