جامعة بيرزيت

BIRZEIT UNIVERSITY

MATHEMATICS DEPARTMENT
MATH330, First Hour Exam

• Name........Key.......... • Number.................... • Section..................

(Q1) (5 points) If

$$f(x) = \frac{x - \sin x}{ln(x+2)}$$

Find $f(\frac{7}{12})$ using four digits rounded.

$$f\left(\frac{7}{12}\right) = f(0.5833) = \frac{0.5833 - 0.5508}{\ln(2.583)} \quad 1$$

$$= \frac{0.03250 \quad 1}{0.9490 \quad 1} = 0.03425 \quad 2$$

(Q2) (5 points) Suppose that $e^h = 1 + h + h^2/2 + h^3/6 + O(h^4)$ and $\sin h = h - h^3/6 + O(h^5)$, find the approximated value of $(e^h \sin h)$ at $h = 0.01$ with its error.

$$e^h \sin h = \left(1 + h + \frac{h^2}{2} + \frac{h^3}{6}\right)\left(h - \frac{h^3}{6}\right) + O(h^4)$$

$$= \underbrace{h + h^2 + \frac{h^3}{3}}_{2} + \underbrace{O(h^4)}_{1}$$

$$e^{0.01} \sin 0.01 = (0.01) + (0.01)^2 + \frac{(0.01)^3}{3} + c \times (0.01)^4$$

$$= \underbrace{0.010100333}_{1} + \underbrace{c \times 10^{-8}}_{1}$$

(Q3) (15 points) Consider the function $g(x) = e^{-x} + \frac{1}{x} + 5$

(a) Show that $g(x)$ has a unique fixed point in the interval $[5, 6]$

(b) Using $p_0 = 5.5$, estimate the fixed point of $g(x)$. Find four iterations.

(c) Find the number of iterations needed to achieve accuracy of 10^{-10}

(a) (1) g is Continuous on $[5, 6]$.

(2) g is decreasing. 1

& $g(5) = 5.2067$ 1 & $g(6) = 5.1696$ 1

$\Rightarrow \quad g(x) \in [5, 6] \quad , \quad \forall x \in [5, 6]$

then there exist at least one fixed point in $[5, 6]$.

For Uniqueness:

(3) $|g'(x)| = \left| -e^{-x} - \frac{1}{x^2} \right| = e^{-x} + \frac{1}{x^2}$ 1 which is decreasing 1

then $|g'(x)| \leq e^{-5} + \frac{1}{5^2} = 0.046737 < 1$ 2

Therefore, there exist a Unique fixed point

(b) $P_0 = 5.5$

$P_1 = 5.18590$, $P_2 = 5.198425$, $P_3 = 5.19789$

$P_4 = 5.19791$. each 1 point.

(c) $k^n \dfrac{|P_1 - P_0|}{1 - k} < 10^{-10}$ 1

$n > 7.15445$ 2

$\therefore \boxed{n = 8}$ 1

(Q4) (6 points) Use the bisection method to estimate the solution of $\sin(\cos x) = x$ in the interval $[0, 1]$. Find 3 iterations.

$f(x) = \sin(\cos x) - x$

a_n	c_n	b_n
0^{+}	0.5	1^{-}
0.5^{+}	0.75	1^{-}
0.5^{+}	0.625	0.75^{-}

each c_i $\div$ (2 points)

(Q5) (5 points) We want to estimate the number $\sqrt[3]{0.008}$ using false position method on $[0.1, 0.3]$. Find one iteration, then find the relative error in this estimation.

Let $f(x) = x^3 - 0.008$.

$$c_0 = 0.3 - \frac{f(0.3)(0.3 - 0.1)}{f(0.3) - f(0.1)}$$

$c_0 = 0.15385$ (2)

Exact value $= 0.2$ (1)

$$\left.\begin{aligned}\text{Relative error} &= \frac{|0.2 - 0.15385|}{|0.2|} \\ &= 0.23075.\end{aligned}\right\} \quad (2)$$

(Q6) (6 points) Approximate π by using Newton's method to find a solution of $sin(x) = 0$, starting with $x_0 = 3$. How many iterations of the method are needed until the solution is accurate to 10^{-5}.

$f(x) = \sin x$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{\sin(p_n)}{\cos(p_n)}$$

$p_0 = 3$

$p_1 = 3.1425465 4$ 2

$p_2 = 3.14159265$ 2

$p_3 = 3.14159265$ 2

(Q7) (6 points) The number p is a simple root for $f(x)$. The table below represents the iterations resulted from estimating p using the secant method. Find R and A.

k	P_k
0	2.5
1	2.151029
2	2.118449
3	2.116785
4	2.116797

2

Since p is a simple root, then $R = 1.618$.

$E_k \approx \lvert p - p_k \rvert$	$\frac{\lvert E_{k+1} \rvert}{\lvert E_k \rvert^{1.618}}$	OR $\lvert p_k - p_{k-1} \rvert$	$\frac{\lvert E_{k+1} \rvert}{\lvert E_k \rvert^{1.618}}$
0.383203	0.1616011	0.348971	0.17894
0.034232	0.38841735	0.03258	0.42384
1.652×10^{-3}	0.38057564.8	0.001664	0.37614
1.2×10^{-5}		0.000012	
2	2		

(Q8) (6 points) Let $f(x) = 3x^2 - e^x - 1$. If the root $p \approx 1.2000030141$, find the multiplicity of the root, then determine the order of convergence and the asymptotic error constant using newton method.

$f(x) = 3x^2 - e^x - 1$

$f'(x) = 6x - e^x$

$f'(p) = 3.879891154 \neq 0$ ② $\Rightarrow$ p is a simple root.

$\Rightarrow \boxed{R = 2}$ ②

$A = \left| \dfrac{f''(p)}{2f'(p)} \right| = 0.345354156$ ②

(Q9) (6 points) The number $p = 0$ is a root for $f(x) = x \sin x$. Use the accelerated Newton's method to estimate this root starting with 1.5. Find three iterations.

$f(x) = x \sin x$

$f'(x) = x\cos x + \sin x \Rightarrow f'(0) = 0$ ①

$f''(x) = -x\sin x + \cos x + \cos x \Rightarrow f''(0) = 2 \neq 0$ ①

$\therefore \boxed{M = 2}$ ①

$$p_{n+1} = p_n - \frac{2 f(p_n)}{f'(p_n)}$$

$p_1 = -1.21156476$ ①

$p_2 = 0.4497639$ ①

$p_3 = -0.015916088$ ①

(Q10) (4 points) Find the fixed points of $g(x) = \frac{x}{2} + \frac{2}{x}$. Then classify them into repelling or attracting.

$$\frac{x}{2} + \frac{2}{x} = x$$

$$2x^2 = x^2 + 4$$

$$x^2 = 4 \Rightarrow x = -2, 2.$$

2

$$g'(x) = \frac{1}{2} - \frac{2}{x^2}$$

$|g'(2)| = 0 < 1 \Rightarrow 2$ is attractive.

$|g'(-2)| = 0 < 1 \Rightarrow -2$ is attractive.

2

(Q11) (6 points) Use Gauss Seidel iteration to solve the system below. Start with $(0,0,0)$, then find two iterations.

$$\begin{aligned} x &= x^2 + y + \cos z \\ y &= x + e^y + z \\ z &= -2x + y + \ln(z+1) \end{aligned}$$

each part is 1 point

$p_1 = g_1(0,0,0) = 1$

$q_1 = g_2(1,0,0) = 2$

$r_1 = g_3(1,2,0) = 0$

$p_2 = g_1(1,2,0) = 4$

$q_2 = g_2(4,2,0) = 11.3890561$

$r_2 = g_3(4, 11.3890561, 0) = 3.389056099.$

(Q12) (10 points) Using Newtons method to estimate the solution of the following system starting with (1, 1), find only one iteration.

$$\begin{cases} \sin x + 2y = 1 \\ 2x + e^{-y} = 0. \end{cases}$$

$f_1(x,y) = \sin x + 2y - 1$, $f_1(1,1) = 1.8$ ①

$f_2(x,y) = 2x + e^{-y}$. $f_2(1,1) = 2.4$ ①

$$J = \begin{pmatrix} \cos x & 2 \\ 2 & -e^{-y} \end{pmatrix} ①, \quad J\Big|_{(1,1)} = \begin{pmatrix} 0.54 & 2 \\ 2 & -0.37 \end{pmatrix} ①$$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{-4.2} \begin{pmatrix} -0.37 & -2 \\ -2 & 0.54 \end{pmatrix} \begin{pmatrix} 1.8 \\ 2.4 \end{pmatrix} \quad ②$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{4.2} \begin{pmatrix} -0.67 + -4.8 \\ -3.6 + 1.3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{4.2} \begin{pmatrix} -5.5 \\ -2.3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1.3 \\ -0.55 \end{pmatrix} \quad ②$$

$$= \begin{pmatrix} -0.3 \\ 0.45 \end{pmatrix} \quad ②$$