BIRZEIT UNIVERSITY

Mathematics Department

Key

Math 330

1st Exam

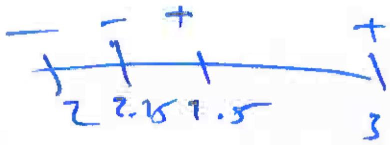
Student name: …………………… ID no.: …… 1st Semester 21/22

sec…….

(problems from 1 to 10, 3 points each)

(1) Use the bisection method to estimate the solution of the equation $x^3 - 7cosx - 19 = 0$, on the interval [2,3].
Find the first three iterations c_0, c_1, c_2 .

$f(2) \simeq -8.09$, $f(3) \simeq 14.93$



1 $c_0 = \frac{2+3}{2} = \underline{2.5}$, $f(2.5) \simeq 2.23$

$\Rightarrow [a_1, b_1] = [2, 2.5]$

1 $c_1 = \frac{2+2.5}{2} = \underline{2.25}$, $f(2.25) \simeq -3.21$

$\Rightarrow [a_2, b_2] = [2.25, 2.5]$

1 $\Rightarrow c_2 = \frac{2.25+2.5}{2} = \underline{2.375}$

(2) Use the False position method to estimate the solution of the equation $x^3 - 7cosx - 19 = 0$, on the interval [2,3]. $= [a_0, b_0]$
Find the first two iterations c_0, c_1 .

1 $c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$

$= 3 - \frac{(14.93)(1)}{14.93 + 8.09}$

1 $\simeq 2.35143$

$f(c_0) \simeq -1.07231 \Rightarrow [a_1, b_1] = [2.35, 3]$

$c_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)}$

$= 3 - \frac{14.93(0.65)}{14.93 + 1.07231}$

1 $\simeq 2.39356$

3) Use Fixed point theorem to show that the function $g(x) = \frac{1}{x^2} + 2$ has a fixed point in the interval [2,3].(existence only)

1 $g(2) = \frac{1}{4} + 2 = 2.25 \in [2,3]$

$g(3) = \frac{1}{9} + 2 = 2.111 \in [2,3]$

1 Since the function is decreasing

$g(3) \leq g(x) \leq g(2)$, for every $x \in [2,3]$

1 $\Rightarrow 2 \leq 2.11 \leq g(x) \leq 2.25 \leq 3$

$\Rightarrow g(x) \in [2,3] \Rightarrow \exists p \in [2,3]$ s.t $g(p) = p$

4) Show why the fixed-point iteration generated by the function $g(x) = \frac{1}{x^2} + 2$ converges in the interval [2,3]

1 $|g'(x)| = \left|\frac{-2}{x^3}\right| = \frac{2}{x^3}$ is also decreasing so 1

1 $\max |g'(x)| \leq \frac{2}{2^3} = \frac{1}{4} < 1 \quad \forall x \in [2,3]$

$\Rightarrow$ FPI converges

(5) Find the fixed points of the function $g(x) = \frac{20}{x-1}$. Show which one is attractive and which one is repulsive and why.

$x = \frac{20}{x-1}$

$x^2 - x - 20 = 0$

$(x-5)(x+4) = 0$

1 $x = 5, \quad x = -4$

$g'(x) = \frac{-20}{(x-1)^2}$

$|g'(x)| = \frac{20}{(x-1)^2}$

$g'(5) = \frac{20}{16} > 1 \Rightarrow x = 5$ is repulsive 1

$g'(-4) = \frac{20}{25} < 1 \Rightarrow x = -4$ is attractive 1

6) The point $p = 2$ is a zero of the function $f(x) = x^3 - 6x^2 + 12x - 8$,
Use Newton iteration to estimate the zero $p = 2$, starting with $p_0 = 2.1$
Find p_1, p_2, p_3

$$p_{n+1} = p_n - \frac{(p_n^3 - 6p_n^2 + 12p_n - 8)}{3p_n^2 - 12p_n + 12}$$

$p_0 = 2.1$

1 $p_1 = 2.066666\ldots$

1 $p_2 = 2.04444\ldots$

1 $p_3 = 2.02[illegible]2\ldots$

7) The point $p = 2$ is a zero of the function $f(x) = x^3 - 6x^2 + 12x - 8$, using Newton iteration to estimate the zero $p = 2$, What is the order of convergence R and the asymptotic error constant A .

$f(x) = (x-2)^3$

1 $M = 3$

1 $R = 1$

1 $A = \frac{2}{3}$

$x^2 - 4x + 4$

$x - 2 \,\overline{)\, x^3 - 6x^2 + 12x - 8}$

$\mp x^3 \pm 2x^2$

$-4x^2 + 12x - 8$

$\pm 4x^2 \mp 8x$

$4x - 8$

$4x - 8$

0

$x^2 - 4x + 4 = (x-2)^2$

8) $p = 4$ is a fixed point of the function $g(x) = \frac{x^2+4}{2x-3}$
Show that the order of convergence of the fixed-point iteration generated by $g(x)$ is at least quadratic .

$$g'(x) = \frac{2x(2x-3) - (x^2+4)(2)}{(2x-3)^2}$$

1

$$= \frac{4x^2 - 6x - 2x^2 - 8}{(2x-3)^2} = \frac{2x^2 - 6x - 8}{(2x-3)^2}$$

2 $g'(4) = \frac{32 - 24 - 8}{25} = \frac{0}{25} = 0$

since $g'(p) = 0 \Rightarrow$ atleast quadratic

9)Consider the following system of equations

$$x = g_1(x,y,z) = 3x^2z - 2y^3$$
$$y = g_2(x,y,z) = xyz$$
$$z = g_3(x,y,z) = 10z + 2xy$$

Use fixed point iteration to find the 1st iteration given that the initial point is (3,2,4)

1 $p_1 = g_1(3,2,4) = 3(3)^2(4) - 2(2)^3 = 92$

1 $q_1 = g_2(3,2,4) = (3)(2)(4) = 24$

1 $r_1 = g_3(3,2,4) = 10(4) + 2(3)(2) = 52$

10)Consider the following system of equations

$$x = g_1(x,y,z) = 3x^2z - 2y^3$$
$$y = g_2(x,y,z) = xyz$$
$$z = g_3(x,y,z) = 10z + 2xy$$

Use Gauss-Sidel iteration to find the 1st iteration given that the initial point is (3,2,4)

1 $p_1 = g_1(3,2,4) = 92$

1 $q_1 = g_2(92,2,4) = (92)(2)(4) = 736$

1 $r_1 = g_3(92,736,4) = 10(4) + 2(92)(736)$
$= 135,464$

The following problems each worth 5 points

11) Solve the following system of equations using L-U factorization and three digits rounding

$$6.33x + 0.113\,y = 6.44$$
$$10.2\,x - 0.212y \;= 9.99$$

$Ax = b$
$LUx = b$
$LY = b$:
then $UX = Y$

$$A = \begin{bmatrix} 6.33 & 0.113 \\ 10.2 & -0.212 \end{bmatrix} \quad m_{21} = \frac{10.2}{6.33} \approx 1.61$$

$$R_2 - 1.61R_1 \begin{bmatrix} 6.33 & 0.113 \\ 0 & -0.394 \end{bmatrix} = U \quad 2 \quad -0.212 - 1.61(0.113) \approx -0.212 - 0.182$$

$$L = \begin{bmatrix} 1 & 0 \\ 1.61 & 1 \end{bmatrix}$$

$$LY = b \Rightarrow \begin{bmatrix} 1 & 0 \\ 1.61 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 6.44 \\ 9.99 \end{bmatrix}$$

$y_1 = 6.44$

$1.61(6.44) + y_2 = 9.99$ 2

$\overset{\approx}{10.4} + y_2 = 9.99$

$y_2 = -0.41$

$UX = Y$

$$\begin{bmatrix} 6.33 & 0.113 \\ 0 & -0.394 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.44 \\ -0.41 \end{bmatrix}$$

$x_2 = \frac{-0.41}{-0.394} \approx 1.04$ 2

$6.33x_1 + 0.113(1.04) = 6.44$

$6.33x_1 = 6.44 - 0.118 = 6.32$

$x_1 = \frac{6.32}{6.33} = 0.998$

12) Find the cost of finding the solution in problem 12 (don't use formulas)

2 Cost To get U: 1 division + 1 multiplication + 1 addition = 3

1 Solving $LY = b \Rightarrow$ 1 multiplication + 1 addition = 2

2 Solving $UX = b \Rightarrow$ 1 division + 1 multiplication + 1 subtraction + 1 division = 4

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13) Use newton method to find the 1^{st} iteration of the following system

$$0 = 3xy - y^3 - x \quad = f_1(x,y)$$
$$0 = 2y^2x - 2x^2 - y \quad = f_2(x,y)$$

given that the initial estimation is (1, 3)

1 $f_1(1,3) = 3(1)(3) - 3^3 - 1 = -19$

$f_2(1,3) = 2(3)^2(1) - 2 - 3 = 13$

1 $J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 3y-1 & 3x-3y^2 \\ 2y^2-4x & 4yx-1 \end{pmatrix}$

1 $J\Big|_{(1,3)} = \begin{pmatrix} 8 & -24 \\ 14 & 11 \end{pmatrix} \quad |J| = 424$

1 $J^{-1} = \frac{1}{424}\begin{pmatrix} 11 & 24 \\ -14 & 8 \end{pmatrix} = \begin{pmatrix} 0.0259 & 0.0566 \\ -0.0330 & 0.0189 \end{pmatrix}$

$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{1}{424}\begin{pmatrix} 11 & 24 \\ -14 & 8 \end{pmatrix}\begin{pmatrix} -19 \\ 13 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{1}{424}\begin{pmatrix} 103 \\ 370 \end{pmatrix} \simeq \begin{pmatrix} 0.757 \\ 2.13 \end{pmatrix}$ 1

14)Estimate the solution of the following equation up to two digits acurracy

$$\frac{200x}{12} = (1 + \frac{x}{12})^{144} - 1 \quad \Rightarrow \frac{200x}{12} + 1 = \left(1 + \frac{x}{12}\right)^{144}$$

,where x is the yearly interest rate

1 $f(x) = 144\ln\left(1 + \frac{x}{12}\right) - \ln\left(\frac{200x}{12} + 1\right)$

1 $f'(x) = 144 \cdot \frac{1}{1+\frac{x}{12}} \cdot \frac{1}{12} - \frac{1}{\frac{200x}{12}+1} \cdot \frac{200}{12}$

$= \frac{144}{x+12} - \frac{200}{200x+12}$

$p_1 = 0.05072555$

$= 0.05124879$

$= 0.051623$

$= 0.051890$

$= 0.0520$

0.052

$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \quad p_0 = 0.05$

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$\Rightarrow \boxed{0.052}$