BIRZEIT UNIVERSITY

Mathematics Department

Math 330

1st Exam

Student name: …………………… ID no.: ……

2nd. Semester 17-18

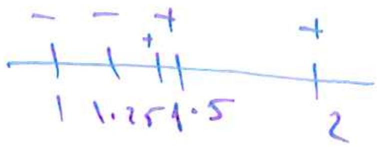
sec……..

(Q# 1)(8 Points) Let $f(x) = x^3 - \ln x - 2$

Estimate the zero of $f(x)$ in [1,2] using the bisection method (do only 4 iterations c_0, c_1, c_2, c_3).and use these iterations to find the order of convergence of the sequence c_n , $n = 1,2,3,....$

$f(1) = -1$

$f(2) > 0$



$c_0 = 1.5$ $f(1.5) > 0$

4

$c_1 = 1.25$ $f(1.25) < 0$

$c_2 = 1.375$ $f(1.375) > 0$

$c_3 = \frac{1.25 + 1.375}{2} = 1.3125$

n	c_n	$\lvert e_n \rvert$	$\left\lvert \frac{e_{n+1}}{e_n} \right\rvert$
0	1.5	—	—
1	1.25	0.25	
2	1.375	0.125	0.5
3	1.3125	0.0625	0.5

2

2

$\downarrow \frac{1}{2}$

Order of Convergence is 1

(Q#2))(15 Points) Consider the fixed point iteration $p_{n+1} = \sqrt[3]{2 + \ln(p_n)} = g(p_n)$.

a- Show that $g(x)$ has a fixed point in I = [1,2].

b- Show that if $p_0 \in I$, then the fixed point iteration converges.

c- Estimate the fixed point p starting with $p_0 = 1.5$, (do only 4 iterations)

(a) $g(x) = (\ln x + 2)^{1/3}$ is increasing (2)

$g(1) = 1.26 \in [1,2]$ (1)

$g(2) = 1.39 \in [1,2]$ (1)

$\Rightarrow g(x) \in [1,2] \quad \forall x \in [1,2]$ (1)

$\Rightarrow$ g has a fixed point in [1,2]

(b) $g'(x) = \frac{1}{3}(\ln x + 2)^{-2/3} \cdot \frac{1}{x}$

$= \frac{1}{3x\sqrt[3]{(\ln x + 2)^2}}$ is decreasing (2)

So $|g'(x)| \le \max_{1 \le x \le 2} |g'(x)| = g'(1)$ (2)

$= \frac{1}{3\sqrt[3]{2.86}} < \frac{1}{3}$ (2)

$\Rightarrow$ The FPI converges $\forall p_0 \in [1,2]$

(c) ~~$P_1 = 1.12014\ldots$~~

~~$P_2 = 1.03647\ldots$~~

~~$P_3 = 1.01180\ldots$~~

~~$P_4 = 1.00$~~

(4) $P_1 = 1.33988\ldots$

$P_2 = 1.31858\ldots$

$P_3 = 1.315506\ldots$

$P_4 = 1.3150560\ldots$

(Q#3) (12 Points) Consider $f(x) = x^3 - \ln x - 2$

a- Start with $p_0 = 1.5$, use Newton iteration to estimate the root of $f(x)$, with $|\text{error}| \le 10^{-5}$

b- Find the order of convergence the above iterations both numerically and theoritically.

$f'(x) = 3x^2 - \frac{1}{x}$

$P_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.3406244$

$P_2 = 1.315584343$

4

$P_3 = 1.31497907$

$P_4 = 1.31497872$

$P_5 = - -$

n	P_n	$\lvert e_n \rvert$	$\frac{\lvert e_{n+1} \rvert}{\lvert e_n \rvert^2}$
0	1.5	—	—
1	1.3406244	0.1693746	
2	1.315584343	0.0250396	0.98674
3	1.3149790	0.00060523	0.96
4	1.3149787	0.00000035	0.955

3

$P = 1.3149787 2$

$f'(p) \neq 0$ (1)

p is a simple root (1) , $R = 2$ (1)

$A = \left| \frac{f''(p)}{2f'(p)} \right| = 0.956$ (2)

0.95

(Q#4) (6 Points) Estimate the solution of the following system using Newton iteration

$$x = \frac{x-\sin(xy)}{3}$$

$$y = \frac{y+\cos(x+y)}{2}$$

Do only one iteration , starting with (2,1)

1.2341795

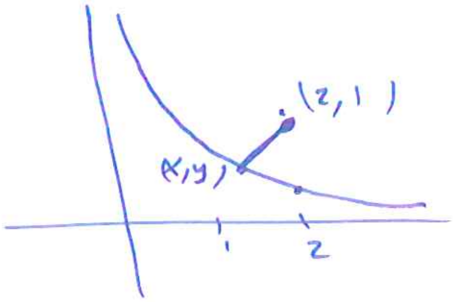
1.235

(Q#5) (6 Points) Find the order of convergence of the fixed point iteration $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$ when p is a simple root

$g'(p) = 0$

(Q#6) (6 Points)

Approximate to within 10^{-2}, the value of x that produces the point on the graph $y = \frac{1}{x}$ that is closest to the point (2,1)



$D^2 = (y-1)^2 + (x-2)^2$

$D^2 = \left(\frac{1}{x} - 1\right)^2 + (x-2)^2$

$2DD' = 2\left(\frac{1}{x} - 1\right)\left(-\frac{1}{x^2}\right) + 2(x-2)$

$0 = -\frac{2}{x^3} + \frac{2}{x^2} + 2x - 4 \quad \cancel{/x^3}$ 3

$f(x) = 2x^4 - 4x^3 + 2x - 2$ 3

$f'(x) = 8x^3 - 12x^2 + 2$

$p_0 = 1.5 \longrightarrow 1.86676$

$= 1.$

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Q1]

c_k	$\lvert E_k \rvert = \lvert c_k - c_{k-1} \rvert$	$\frac{\lvert E_{k+1} \rvert}{\lvert E_k \rvert}$
1.5	—	
1.25	0.25	0.5
1.375	0.125	0.5
1.3125	0.0625	

4 2

$R = 1$, $A = 0.5$

2

Q2 $g(x) = \sqrt[3]{2+\ln x} = (2+\ln x)^{\frac{1}{3}}$

(a) (1) g is cont. in $[1,2]$

(2) g is increasing in $[1,2]$

2

$\Rightarrow$ $g(1) \le g(x) \le g(2)$

1 1

$1 < 1.25992 = \sqrt[3]{2} \le g(x) < \sqrt[3]{2+\ln 2} \doteq 1.39129 < 2$

$\Rightarrow$ g has Fixed pts in $[1,2]$ 1

(b) $g'(x) = \frac{1}{3}(2+\ln x)^{-2/3} \cdot \frac{1}{x}$

2

$|g'(x)| = \dfrac{1}{3x\sqrt[3]{(2+\ln x)^2}}$ which is decreasing

2 2

$\Rightarrow |g'(x)| \le \dfrac{1}{3(1)\sqrt[3]{(2+\ln 1)^2}} = 0.20998 < 1$

$\Rightarrow$ FPI converges.

(c) $P_0 = 1.5$.

$P_1 = 1.33988$

$P_2 = 1.31859$

$P_3 = 1.31551$

$P_4 = 1.31506$

4

Q3) $f(x) = x^3 - \ln x - 2$, $f' = 3x^2 - \frac{1}{x}$, $f'' = 6x + \frac{1}{x^2}$

(a)

4

P_k	$\lvert P_k - P_{k-1} \rvert$
1.5	——
1.340624401	0.159375599
1.315584343	0.025040058
1.31497907	0.000605273
1.31497872	0.00000035

1

(b) Theoretically: $p = 1.31497872$, $f'(p) = 4.427038522 \neq 0$

$\Rightarrow p$ is simple root. 1

$\Rightarrow R = 2$, $A = \left| \frac{f''(p)}{2f'(p)} \right| = 0.956416432$

1 2

Numerically:

$\lvert E_k \rvert$	$\frac{\lvert E_{k+1} \rvert}{\lvert E_k \rvert^2}$
0.159375599	0.985806484
0.025040058	0.965340752
0.000605273	0.955356453
0.000000035	

3

Q4 $\quad f_1 = x - \frac{x}{3} + \frac{\sin(xy)}{3} = \frac{2x}{3} + \frac{\sin(xy)}{3}$

$$f_2 = y - \frac{y}{2} - \frac{\cos(x+y)}{2} = \frac{y}{2} - \frac{\cos(x+y)}{2}$$

$$f_1(2,1) = \frac{4}{3} + \frac{\sin(2)}{3} = 1.636$$

$$f_2(2,1) = \frac{1}{2} - \frac{\cos(3)}{2} = 0.995.$$

$$J = \begin{bmatrix} \frac{2}{3} + \frac{y\cos(xy)}{3} & \frac{x\cos(xy)}{3} \\ \frac{\sin(x+y)}{2} & \frac{1}{2} + \frac{\sin(x+y)}{2} \end{bmatrix}$$

$$J\Big|_{(2,1)} = \begin{bmatrix} 0.528 & -0.277 \\ 0.071 & 0.571 \end{bmatrix}, \quad |J| = 0.321$$

$$J^{-1}\Big|_{(2,1)} = \begin{bmatrix} 1.779 & 0.863 \\ -0.221 & 1.645 \end{bmatrix}$$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1.779 & 0.863 \\ -0.221 & 1.645 \end{pmatrix}\begin{pmatrix} 1.636 \\ 0.995 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3.769 \\ 1.275 \end{pmatrix} = \begin{pmatrix} -1.769 \\ -0.275 \end{pmatrix}$$

4

$x = \dfrac{x - \sin(xy)}{3} \Rightarrow 2x + \sin(xy) = 0 = f_1(x, y)$

$y = \dfrac{y + \cos(x+y)}{2} \Rightarrow y - \cos(x+y) = 0 = f_2(x, y)$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} - J^{-1}\Big|_{(p_0, q_0)} \cdot \begin{pmatrix} f_1(p_0, q_0) \\ f_2(p_0, q_0) \end{pmatrix} \quad 1$$

$f_1(2,1) = 4.909$ 1, $f_2(2,1) = 1.990$ 1

$$J = \begin{pmatrix} 2 + y\cos(xy) & x\cos(xy) \\ \sin(x+y) & 1 + \sin(x+y) \end{pmatrix}$$

$$J\Big|_{(2,1)} = \begin{pmatrix} 1.584 & -0.832 \\ 0.141 & 1.141 \end{pmatrix} \quad 1$$

$|J| = \cancel{0.49}$
$= 1.925$
$\cancel{|J|} =$

$$J^{-1}\Big|_{(2,1)} = \begin{pmatrix} 0.593 & 0.433 \\ -0.073 & 0.824 \end{pmatrix} \quad 1$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0.593 & 0.433 \\ -0.073 & 0.824 \end{pmatrix} \begin{pmatrix} 4.909 \\ 1.990 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3.773 \\ 1.28 \end{pmatrix} = \begin{pmatrix} -1.773 \\ -0.28 \end{pmatrix} \quad 1$$

Q5 method one:

Apply Taylor's Expansion of $f(x)$ about $x = P_n$

$$f(x) = f(P_n) + f'(P_n)(x-P_n) + \frac{f''(c)}{2}(x-P_n)^2$$

$$\cancel{f(P)}^{0} = f(P_n) + f'(P_n)(P-P_n) + \frac{f''(c)}{2}(P-P_n)^2$$

$$\div f'(P_n) \Rightarrow 0 = \frac{f(P_n)}{f'(P_n)} + P - P_n + \frac{f''(c)}{2f'(P_n)}(P-P_n)^2$$

$$\Rightarrow \quad P - \left(P_n - \frac{f(P_n)}{f'(P_n)}\right) = \frac{-f''(c)}{2f'(P_n)}(P-P_n)^2$$

$$\left|\frac{P - P_{n+1}}{(P-P_n)^2}\right| = \left|-\frac{f''(c)}{2f'(P_n)}\right|$$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^2} = \left|\frac{f''(p)}{2f'(p)}\right| \Rightarrow R = 2$$

$f(p) = 0$

method 2: $\quad g(x) = x - \dfrac{f(x)}{f'(x)}$

(1) $$g'(x) = 1 - \frac{(f')^2 - ff''}{(f')^2} \Rightarrow g'(p) = 1 - \frac{(f'(p))^2 - 0}{(f'(p))^2}$$

$$= 1 - 1 = 0$$

$$g'(x) = 1 - 1 + \frac{ff''}{(f')^2} = \frac{ff''}{(f')^2}$$

$$g''(x) = \frac{(f')^2[ff''' + f''f'] - ff''(2)f'f''}{(f')^4} \Rightarrow g''(p) = \frac{f''(p)}{f'(p)} \neq 0$$

$$\Rightarrow R = 2$$

Q6

$$d = \sqrt{(2-x)^2 + \left(1-\frac{1}{x}\right)^2}$$

$$d^2 = (2-x)^2 + \left(1-\frac{1}{x}\right)^2$$

$$2dd' = -2(2-x) + 2\left(1-\frac{1}{x}\right)\left(\frac{1}{x^2}\right) = 0$$

$$-2x^3(2-x) + 2x\left(1-\frac{1}{x}\right) = 0$$

$$-4x^3 + 2x^4 + 2x - 2 = 0 = f(x)$$ 3

$$f' = -12x^2 + 8x^3 + 2$$

$f(1) < 0$, $f(2) > 0$

take $p_0 = 1.5$ use Newton

p_k	$\lvert p_k - p_{k-1} \rvert$
1.5	
2.6875	0.18
2.26174	0.42
2.00309	0.258
1.8898	0.12
1.86756	0.022
$\boxed{1.86676}$	0.001

3