KEY

MATHEMATICS DEPARTMENT
Math330 -First Exam-
Second Semester 2018/2019

• Name.................................. • Number......................... • Section.........................

Q1) [3 points] If $f(h) = \frac{1}{2}h + h^2 + O(h^4)$ and $g(h) = 2h + h^3 + O(h^5)$, estimate $(fg)(0.1)$

$(fg)(h) = h^2 + 2h^3 + O(h^4)$ (2)

$(fg)(0.1) \approx (0.1)^2 + 2(0.1)^3 = 0.012$ (1)

Q2) [~~3~~ 4 points] Use Gauss-Seidel iteration to find (p_2, q_2) for the system below using $(p_0, q_0) = (2, 0)$

$$x = \frac{1}{4}x^2 + xy$$

$$y = \frac{1}{x} + 2y - 1$$

$p_1 = g_1(2,0) = 1$

$q_1 = g_2(1,0) = 0$ (4)

$p_2 = g_1(1,0) = \frac{1}{4}$

$q_2 = g_2(\frac{1}{4},0) = 3$

Q3) [3 points] Estimate the fixed point of $g(x) = 0.5x^2 + \frac{1}{x} - 0.5$ using $p_0 = 1.3$ with three iterations.

$p_1 = 1.114230769$ (3)

$p_2 = 1.018235255$

$p_3 = 1.000492832$

Q4) [4 points] Consider the function $f(x) = \ln(x-1) + \sin(x-2)$ with the root $p = 2$. Find, theoretically, the order of convergence and the asymptotic error constant if we used Newton's method to estimate this root.

$f' = \frac{1}{x-1} + \cos(x-2) \Rightarrow f'(2) = 2 \neq 0$

$\Rightarrow M = 1$ (1) & $R = 2$ (1)

$f'' = \frac{-1}{(x-1)^2} - \sin(x-2)$

$A = \left| \frac{f''(2)}{2f'(1)} \right| = \left| \frac{-1}{2(2)} \right| = \frac{1}{4}$ (1) (1)

$f''(2) = -1$

Q5) [4 points] Consider the function $g(x) = \sqrt{x+8}$. Show that $g(x)$ has a fixed point in $[3, 4]$

g is increasing (2)

$\Rightarrow \quad g(3) \leq g(x) \leq g(4)$

$3 < 3.31 \leq g(x) \leq 3.46 < 4$ (2)

Q6) [4 points] Let A be a 3×3 matrix. Find the cost of evaluating $A - A^2$ (1)

(2)

Cost = cost of A^2 + cost of subtraction

$= 2(3)^3 - 3^2 + 9$

$= 45 + 9$

$= 54$ (1)

Q7) [4 points] Estimate the root of $f(x) = x^4 - 10$ in the interval $[1, 2]$ using the false position method. Find two iterations.

$$c_0 = 2 - \frac{f(2)(2-1)}{f(2)-f(1)} = 2 - \frac{6}{6--9} = \boxed{1.6} \quad (2)$$

$$f(1.6) = -3.4464$$

$$c_1 = 2 - \frac{f(2)(2-1.6)}{f(2)-f(1.6)} = 2 - \frac{6(0.4)}{6--3.4464} = \boxed{1.7459} \quad (2)$$

Q8) [4 points] Consider the function $g(x) = \frac{1}{18}(27x - x^3)$ with the fixed point $p = 3$.
Find, theoretically, the order of convergence if we used the fixed point iteration to estimate $p = 3$

$$g' = \frac{1}{18}[27 - 3x^2] \Rightarrow g'(3) = 0 \quad (2)$$

$$g'' = \frac{1}{18}[-6x] = \frac{-x}{3} \Rightarrow g''(3) = -1 \quad (1)$$

$$\Rightarrow R = 2 \quad (1)$$

Q9) [4 points] Find the fixed points of $g(x) = \dfrac{12}{7 - x}$, then classify them into repulsive or attractive.

$$x = \frac{12}{7-x}$$

$$7x - x^2 = 12$$

$$x^2 - 7x + 12 = 0$$

$$(x-3)(x-4) = 0$$

$$x = 3, 4 \quad (2)$$

$$g' = \frac{12}{(7-x)^2}$$

$$|g'(3)| = \left|\frac{12}{16}\right| = \frac{3}{4} < 1 \text{ attractive } (1)$$

$$|g'(4)| = \left|\frac{12}{9}\right| = \frac{4}{3} > 1 \text{ repulsive. } (1)$$

Q10) [6 points] Consider the function $f(x) = x^3 - 3x^2 + 3x - 1 = (x-1)^3$ with Newton's method.

(a) Find, theoretically, the order of convergence and asymptotic error constant of the root $p = 1$

(b) Prove your claims in part (b), numerically, using $p_0 = 1.5$ with three iterations of Newton's method.

(a) $M = 3$

$\Rightarrow \alpha = 1$, $A = \frac{3-1}{3} = \frac{2}{3}$

(1) (1)

(b)

p_n	$\lvert E_n \rvert = \lvert p - p_n \rvert$	$\lvert E_{n+1} \rvert / \lvert E_n \rvert$
1.5	0.5	—
1.333333333	0.333333333	2/3
1.222222222	0.222222222	2/3
1.148148148	0.148148148	2/3

$\downarrow$ 2/3

(2) (2)

OR

p_n	$\lvert E_n \rvert \simeq \lvert p_n - p_{n-1} \rvert$	$\lvert E_{n+1} \rvert / \lvert E_n \rvert$
1.5	—	—
1.333333333	0.166666667	—
1.222222222	0.111111111	0.666666664
1.148148148	0.074074074	0.666666666

$\downarrow$ 2/3

Q11) [~~5~~ 6 points] Given the system:

$$\begin{aligned} xy^2 + 2x^2 + 2 &= 0 \\ x^2y + y^2 - 1 &= 0 \end{aligned}$$

Using Newton's method with $(p_0, q_0) = (0, 1)$, find (p_1, q_1).

$$J = \begin{bmatrix} y^2+4x & 2xy \\ 2xy & x^2+2y \end{bmatrix} \quad (1)$$

$$f_1(0,1) = 2, \quad f_2(0,1) = 0 \quad (2)$$

$$J\big|_{(0,1)} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad (1) \quad \det(J) = 2$$

$$J^{-1}\big|_{(0,1)} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad (1)$$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (1)$$

7

Q12) [~~6~~ points] Consider the system:
$$\begin{aligned} 0.84\,x &+ 1.2\,y &= 3.2 \\ 1.5\,x &+ 3\,y &= 4.5 \end{aligned}$$

(a) Use Gauss-Jordan reduction with 2-digit rounding to solve the above system.

(b) Find the cost of part (a).

(a) $\left[\begin{array}{cc|c} 0.84 & 1.2 & 3.2 \\ 1.5 & 3 & 4.5 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 1.4 & 3.8 \\ 1.5 & 3 & 4.5 \end{array}\right]$ (1)

$R_2 - 1.5R_1$ $\left[\begin{array}{cc|c} 1 & 1.4 & 3.8 \\ 0 & 0.9 & -1.2 \end{array}\right]$ (1)

$\left[\begin{array}{cc|c} 1 & 1.4 & 3.8 \\ 0 & 1 & -1.3 \end{array}\right]$ (1)

$R_1 - 1.4R_2$ $\left[\begin{array}{cc|c} 1 & 0 & 5.6 \\ 0 & 1 & -1.3 \end{array}\right]$ (1) $\Rightarrow$ $x = 5.6$ & $y = -1.3$

(b)

Step	±	×	÷
1	1(2)	1(2)	2
2	1(1)	1(1)	1
	3	3	3

Total cost = 9 (3)