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Birzeit University-Dept. of Mathematics
Maec330-Microeconomic Theory

First Hour Exam (Instructor: Dr. Marwan Aloqeili) Spring 2010/2011

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Question 1 Answer by TRUE or FALSE:

1. Let A be a 4×4 symmetric matrix and let A_k denotes a kth order leading principal minor of A. Answer questions (a)-(e) below:

   (a) If $A_1 < 0$, $A_2 > 0$, $A_3 < 0$ and $A_4 < 0$ then A is a positive definite matrix. (....F......)

   (b) If $A_1 = 0$, $A_2 > 0$, $A_3 > 0$ and $A_4 = 0$ then A is either positive semidefinite or indefinite. (.....F)

   (c) If $A_2 < 0$ then A is indefinite. (.....T.....)

   (d) A has 15 leading principal minors. (.....F.....)

   (e) If $A_1 = 0$, $A_2 > 0$, $A_3 = 0$ and $A_4 > 0$ then A might be positive semidefinite or negative semidefinite. (.....T)

2. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, U is open and f is of class C^2, $Df(x^*) = 0$, $x^* \in U$. Answer questions (i)-(v) below:

   (i) If f has a local minimum at the point x^* then $D^2f(x^*)$ is positive definite. (.....T.....)

   Semi

   (ii) If $D^2f(x^*)$ is positive semidefinite then f has global maximum at the point x^*. (...F......)

   (iii) If $D^2f(x^*)$ is indefinite then f does not have a local extreme value at x^*. (.....T.....)

   (iv) If $D^2f(x)$ is positive semidefinite at every point $x \in U$ then f has a global minimum at x^*. (.....T.......)

   (v) If $D^2f(x^*) = 0$ then the second order test fails. (.....T.....)

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Question 2 A firm produces some good according to the function $q = k^{1/4}l^{1/4}$. Suppose that the price of q is p and the prices of k and l are r and w, respectively. Define the profit function $\pi(k,l)$. Find the quantities of K and L that maximize profit. Show that $D^2\pi(k,l)$ is negative definite.

~~$\pi(k,l) = R(k,l) - C(k,l)$~~ $\quad P(k,l) =$

π = Revenue - Cost.

$\pi(k,l) = kr + Lw - p(k^{1/4} l^{1/4})$

(1) $\frac{d\pi}{dk} = r - pL^{1/4}\frac{1}{4}k^{-3/4} = 0$

(2) $\frac{d\pi}{dL} = w - p\,k^{\frac{1}{4}}\frac{1}{4}L^{-3/4} = 0$

$(1)(2) \Rightarrow p = \frac{r}{\frac{1}{4}L^{1/4}k^{-3/4}} = \frac{w}{\frac{1}{4}k^{1/4}L^{-3/4}}$

$\frac{r(k^{1/4}L^{-3/4})}{L^{-3/4}k^{-3/4}} = \frac{w(L^{1/4}k^{-3/4})}{L^{-3/4}k^{-3/4}}$

$rk = wL$

$k = \frac{w}{r}L \Rightarrow L = \frac{r}{w}k$

(1) $\Rightarrow r - \frac{1}{4}pL^{1/4}k^{-3/4} = 0$

$r = \frac{1}{4}p\left(\frac{r}{w}k\right)^{1/4}k^{-3/4}$

$\frac{4r}{p} = \left(\frac{r}{w}\right)^{\frac{1}{4}}k^{\frac{1}{4}}k^{-3/4}$

$\frac{4r}{p} = \left(\frac{r}{w}\right)^{\frac{1}{4}}k^{-\frac{1}{2}}$

$\frac{4r}{p}\frac{w^{\frac{1}{4}}}{r^{\frac{1}{4}}} = \frac{1}{\sqrt{k}}$

$\Rightarrow \frac{1}{k} = \frac{16r^{\frac{3}{4}\cdot 2}\sqrt{w}}{p^2}$

$\boxed{k = \frac{p^2}{16r^{3/4}w^{\frac{1}{2}}}}$

(2) $\Rightarrow w = \frac{1}{4}pk^{\frac{1}{4}}L^{-3/4}$

$w = \frac{1}{4}p\left(\frac{p^2}{16r^{3/4}w^{\frac{1}{2}}}\right)^{\frac{1}{4}}L^{-3/4}$

$\left(\frac{4w(16r^{3/4}w^{\frac{1}{2}})^{\frac{1}{4}}}{p\,(p^2)^{\frac{1}{4}}}\right)^4 = (L^{-3/4})^4$

$L^{-3} = (4w)^4(16r^{3/4}w^{\frac{1}{2}})$

$D^2\pi(k,l) = \begin{vmatrix} -\frac{1}{4}pL^{1/4}k^{-7/4}\left(-\frac{3}{4}\right) & -\frac{1}{4}\frac{1}{4}pL^{-3/4}k^{-3/4} \\ \left(-\frac{1}{4}\right)\left(\frac{1}{4}\right)pL^{-3/4}k^{-3/4} & -\frac{1}{4}\left(-\frac{3}{4}\right)pk^{\frac{1}{4}}L^{-7/4} \end{vmatrix}$

$\frac{3}{16}p$

$D^2\pi(k,l) = \begin{vmatrix} \frac{d^2\pi}{dk^2} & \frac{d^2\pi}{dkdL} \\ \frac{d^2\pi}{dkdL} & \frac{d^2\pi}{dL^2} \end{vmatrix}$

$\frac{d^2\pi}{dk^2} < 0$

$\left(\frac{d^2\pi}{dk^2}\right)\left(\frac{d^2\pi}{dL^2}\right) - \left(\frac{d^2\pi}{dkdL}\right)^2 > 0$

$\Rightarrow$ negative def

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Question 3 Use the Lagrange multiplier method to find the minimum of the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraints $x + y = 9$ and $2y + 2z = 3$. Explain the problem geometrically and conclude that a minimum always exists.

$$Dh = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{Rank} = 2$$

2 boundryes constraints.

$$L(x,y,z) = x^2+y^2+z^2 - \mu_1(x+y-9) - \mu_2(2y+2z-3)$$

$$\frac{\partial L}{\partial x} = 2x - \mu_1 = 0$$

$$\frac{\partial L}{\partial y} = 2y - \mu_1 - 2\mu_2 = 0$$

$$\frac{\partial L}{\partial z} = 2z - 2\mu_2 = 0$$

$x + y = 9$

$2y + 2z = 3$

$\mu_1 = 2x = 2y - 2\mu_2$

$\Rightarrow x = y - \mu_2$ ①

$\mu_2 = y - x$

③ $\Rightarrow \mu_2 = \frac{2z}{2} = z$

$\Rightarrow z = y - x$

④ $\Rightarrow 9 = y + x$

$z + 9 = 2y \Rightarrow z = 2y - 9$

⑤ $\Rightarrow 2y + 2(2y-9) = 3$

$2y + 4y - 18 = 3$

$\cancel{2}6y = \cancel{21}\ 7$

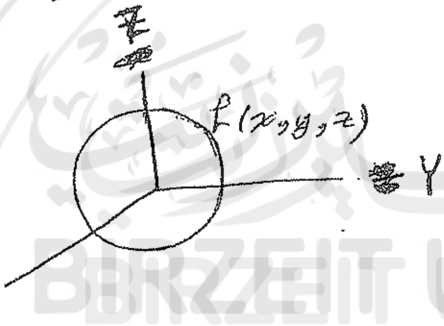
$\boxed{y = 7/2}$

$x + y = 9$

$x = 9 - 7/2 = \boxed{\frac{11}{2} = x}$

$2y + 2z = 3$

$7 + 2z = 3 \Rightarrow z = -\frac{4}{2} = \boxed{-2}$



min always exists since.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 \end{pmatrix} = H$$

$m = 2 \qquad n = 3$

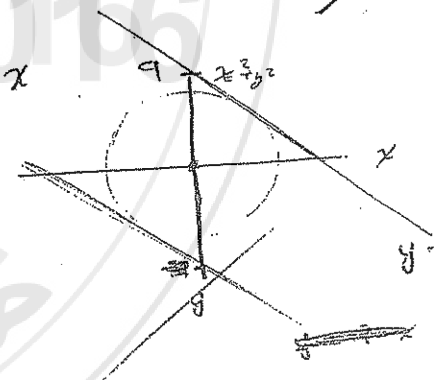
$(-1)^n = -1 \qquad (-1)^m = 1$

$\det(H) > 0$ the sam sighn of $(-1)^m$

so its postive def.

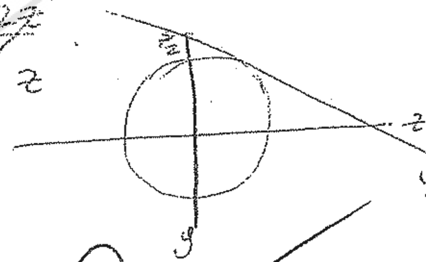
$\Rightarrow$ min is exist

$y = 9 - x$



$y = \frac{3-2z}{2}$

$y = \frac{3}{2} - z$



$f(x,y,z)$ is increasing

$\Rightarrow x^2 + y^2 + z^2 \geq 0$

$\Rightarrow$ critical is the min.

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Question 4 Find the maximum and the minimum of the function $x^2 + y^2$ subject to the constraint $x^2 + y^2 + xy = 1$.

$\Rightarrow (2x+y \quad 2y+x)$ $\quad m=1,\ n=2,\ n-m=1$

$L(x,y) = x^2+y^2 - \mu(x^2+y^2+xy-1)$

$\frac{dL}{dx} = 2x - \mu 2x - \mu y = 0$ $\quad \cancel{\mu = \frac{2x}{2x}}$

$\frac{dL}{dy} = 2y - \mu 2y - \mu x = 0$ $\quad \cancel{\mu = \frac{2y}{2y}}$ $\quad \cancel{\Rightarrow 2y = 2x}$

$x^2+y^2+xy = 1$

$2x = \mu(2x+y)$

$\mu = \frac{2x}{2x+y}$

$\mu = \frac{2y}{2y+x}$

$\frac{2y}{2y+x} = \frac{2x}{2x+y}$

$x(2y+x) = y(2x+y)$

$\cancel{2xy} + x^2 = \cancel{2xy} + y^2$

$x^2 = y^2$

$x = \pm y$

$x = y$ $\quad (3) \Rightarrow x^2 + x^2 + x^2 = 1$

$3x^2 = 1$

$x^2 = \frac{1}{3}$

$x = \pm \frac{1}{3} \Rightarrow y = \pm\frac{1}{3}$

$x = \frac{1}{3} \quad y = \frac{1}{3}$

$x = -y \Rightarrow x^2 + \cancel{y^2} - \cancel{x^2} = 1$

$x^2 = 1$

$x = \pm 1$

$\Rightarrow x = 1 \Rightarrow y = -1$

$x = -1 \Rightarrow y = 1$

$(\frac{1}{3}, \frac{1}{3})\ (-\frac{1}{3}, -\frac{1}{3})\ (1,-1)\ (-1,1)$ are criticals.

$\cancel{H = \begin{pmatrix} 0 & 2x & 2y \\ 2x \\ 2y \end{pmatrix}}$ $\quad L_x = 2x \quad L_y = 2y$

$$H = \begin{pmatrix} 0 & 2x+y & 2y+x \\ 2x+y & 2 & 0 \\ 2y+x & 0 & 2 \end{pmatrix}$$

$$H_{(\frac{1}{3},\frac{1}{3})} = \begin{pmatrix} 0 & \frac{2}{3}+\frac{1}{3} & \frac{2}{3}+\frac{1}{3} \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$|H| = -1(2) + 1(-2) = -4 < 0$

$(-1)^m = -1$. positive def. $\Rightarrow$ min.

$$H_{(\frac{1}{3},\frac{1}{3})} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

$|H| = (-1)(2) - (1)(2)$
$= -2 - 2 = -4 < 0$
$\Rightarrow$ min

$$H_{(1,-1)} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

$|H| = (-1)(2) + (-1)(2) = -2 - 2 = -4 < 0$
min.

$$H_{(-1,+1)} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$|H| = (+1)(-2) + 1(-2)$
$= -2 - 2 = -4 < 0$

max ?

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Question 5 Maximize the function $f(x,y) = x + y$ subject to the constraints $x^2 + y^2 \le 1$ | $x + y \ge 0$. $\Rightarrow -(x+y) \le 0$

$m = 2$
$n = 2$

$\begin{pmatrix} 2x & 2y \\ 1 & 1 \end{pmatrix}$

Rank = 1
one bounding constraint.

(a) Define the Lagrangian and solve the first order conditions.

$L(x,y) = x + y - \lambda_1(x^2+y^2-1) + \lambda_2(x+y)$

$\frac{dL}{dx} = 1 - 2x\lambda_1 + \lambda_2 = 0$

$\frac{dL}{dy} = 1 - 2y\lambda_1 + \lambda_2 = 0$

$\Rightarrow \lambda_2 = 2x\lambda_1 - 1 = 2y\lambda_1 - 1$

$2x\lambda_1 = 2y\lambda_1$

$x = y$

$\lambda_1 > 0$

$x^2+y^2 \le 1$

$x+y \ge 0$

$\lambda_1 \ge 0$

$\lambda_2 \ge 0$

$\lambda_1(x^2+y^2-1) = 0$

$\lambda_2(x+y) = 0$

if $\lambda_1 = 0 \Rightarrow \lambda_2 = -1$ X

$\Rightarrow \lambda_1 > 0 \Rightarrow x^2+y^2-1 = 0$

$2x^2 = 1$

$x^2 = \frac{1}{2} \Rightarrow x = +\sqrt{\frac{1}{2}} \Rightarrow y = \sqrt{}$

$x = -\sqrt{\frac{1}{2}} \Rightarrow y = -\sqrt{}$

criticals $\Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$H = \begin{vmatrix} 0 & 0 & 2x & 2y \\ 0 & 0 & 1 & 1 \\ 2x & 1 & 0 & 0 \\ 2y & 1 & 0 & 0 \end{vmatrix}$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{-2}{\sqrt{2}} = -\sqrt{}$

$\Rightarrow 2\lambda_1 x = 1 + \lambda_2 > 0$

$2\lambda_1 y = 1 + \lambda_2 > 0$

$\Rightarrow \lambda_2 > 0$

$x^2+y^2 = 1$

$x > 0 \Rightarrow x+y > 0$

$y > 0$

$\lambda_2 = 0$

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(b) Check the second order conditions to find the maximizer.

$$H = \begin{pmatrix} 0 & 0 & 2x & 2y \\ 0 & 0 & 1 & 1 \\ 2x & 1 & 0 & 0 \\ 2y & 1 & 0 & 0 \end{pmatrix}$$

$\det(H) = 0$
test fails.

(c) Use the envelope theorem to estimate the maximum value of the function $f(x, y) = 1.1x + y$ subject to $x^2 + y^2 \leq 1$, $x + y \geq 0$.

$\frac{df}{da} = x \qquad da = \frac{1}{10}$

$df = \frac{df}{da} \cdot da = x \cdot \frac{1}{10} \Rightarrow \frac{1}{\sqrt{2}} \cdot \frac{1}{10} = \frac{1}{10\sqrt{2}}$

$f_{new} = \sqrt{2} + \frac{1}{10\sqrt{2}} \Rightarrow \frac{10\sqrt{2}\sqrt{2}+1}{10\sqrt{2}} = \frac{20+1}{10\sqrt{2}} = \frac{21}{10\sqrt{2}}$

Good Luck