~~Q~~ / Q(1) a, b

32/50 +3

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Birzeit University-Dept. of Mathematics
Maec330-Microeconomic Theory

First Hour Exam (Instructor: Dr. Marwan Aloqeili) Spring 2011/2012
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Question 1 Prove the following:

(a) Let $u \in \mathbb{R}^n$. Show that the matrix $A = uu^t$ is symmetric and positive semidefinite. (Hint: use the fact that $v^t v = v.v = |v|^2$, $v \in \mathbb{R}^n$).

~~Let $u \in \mathbb{R}^n \Rightarrow u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n \Rightarrow u^T = (u_1, \dots, u_n)$~~

~~$\Rightarrow uu^T = u_{11} + u_{22} + \dots + u_{nn} = A$~~

صفر

(b) Show that if A is a symmetric $n \times n$ positive definite matrix and λ is an eigenvalue of A then $\lambda > 0$. Recall that λ is and eigenvalue of A if there exists a nonzero vector $x \in \mathbb{R}^n$ such that $Ax = \lambda x$.

~~Let A is a symmetric and +ve def. and let $x \in \mathbb{R}^n$~~

~~$\Rightarrow x^T A x$ is also symmetric and +ve def.~~

~~$\Rightarrow Ax - \lambda x = 0 \Rightarrow (A - \lambda I)x = 0$~~

~~$\Rightarrow |A - \lambda I| = 0$~~

$\Rightarrow$

صفر

(c) Show that if A is a positive definite matrix then $a_{ii} > 0$. Assume A $n \times n$ matrix.

Let A is +ve definite $\Rightarrow$ all L.P.Ms > 0. (Thm)

$\Rightarrow 1^{st}$ LPMs $= |a_{11}|, |a_{22}|, |a_{33}| \dots, |a_{ii}| > 0$.

$\Rightarrow a_{ii} > 0 \quad \forall \; i = 1, 2, \dots, n$.

zero

Question 2 A firm produces a certain good according to the function $f(x, y, z) = \ln(x+1) + \ln(y+1) + \ln(z+1)$. Suppose that the price of the output is p and the prices of the inputs x, y and z are r, s and w, respectively. Find the input quantities that maximize profit. Check the second order conditions.

$$f(x,y,z) = \ln(x+1) + \ln(y+1) + \ln(z+1)$$

$$\Rightarrow R(x,y,z) = p\left(\ln(x+1) + \ln(y+1) + \ln(z+1)\right) \qquad R: \text{revenue.}$$

$$\Rightarrow C(x,y,z) = xr + ys + zw \quad :$$

$$\pi = R(x,y,z) - C(x,y,z)$$

$$= p(\ln(x+1) + p\ln(y+1) + p\ln(z+1) - xr - y\cdot s + zw$$

$$\Rightarrow \frac{\partial \pi}{\partial x} = \frac{p}{x+1} - r = 0 \quad --- \quad (1)$$

$$\frac{\partial \pi}{\partial y} = \frac{p}{y+1} - s = 0 \quad --- \quad (2)$$

$$\frac{\partial \pi}{\partial z} = \frac{p}{z+1} - w = 0 \quad --- \quad (3)$$

From

$$(1) \Rightarrow \frac{p}{x+1} = r \Rightarrow x+1 = \frac{1}{pr} \Rightarrow x = \frac{1}{pr} - 1$$

$$(2) \Rightarrow \frac{p}{y+1} = s \Rightarrow y+1 = \frac{1}{ps} \Rightarrow y = \frac{1}{ps} - 1$$

$$(3) \Rightarrow \quad - - - - - \quad z = \frac{1}{pw} - 1 \quad \Rightarrow$$

since $(x, y, z) \in R^3_{+++}$

$$\Rightarrow \frac{1}{pr} - 1 > 0 \;,\; \frac{1}{ps} - 1 > 0 \;,\; \frac{1}{pw} - 1 > 0 \quad --$$

$$\Rightarrow \frac{1}{pr} > 1 \;,\; \frac{1}{ps} > 1 \;,\; \frac{1}{pw} > 1$$

$p, r, s, w > 0$ (Prices)

$$\Rightarrow 1 > pr \;,\; 1 > ps \;,\; 1 > pw$$

8

$$Dh(x,y,z) = \begin{pmatrix} 1 & 1 & 1 \\ 2x & 2y & 1 \end{pmatrix} \Rightarrow \text{Rank} = m = 2$$

$\Rightarrow$ NDCQ satisfied.

Question 3 Use the Lagrange multiplier method to find the maximum of the function $f(x,y,z) = x + 2z$ subject to the constraints $x + y + z = 1$ and $x^2 + y^2 + z = 7/4$. Follow the steps below:

(a) Define the Lagrangian and write down the set of first order conditions.

~~$L(x,y,z,\mu_1,\mu_2) = f(x,y,z) - \mu_1(x+y+z) - \mu_2(x^2+y^2+z)$~~

$$L(x,y,z,\mu_1,\mu_2) = x + 2z - \mu_1(x+y+z-1) - \mu_2\left(x^2+y^2+z-\frac{7}{4}\right)$$

① $\frac{\partial L}{\partial x} = 1 - \mu_1 - 2\mu_2 x = 0$, ② $\frac{\partial L}{\partial y} = -\mu_1 - 2\mu_2 y = 0$

③ $\frac{\partial L}{\partial z} = 2 - \mu_1 - \mu_2 = 0$ ④ $\frac{\partial L}{\partial \mu_1} = -(x+y+z-1) = 0$

⑤ $\frac{\partial L}{\partial \mu_2} = -\left(x^2+y^2+z-\frac{7}{4}\right)$

(b) Write x and y in terms of μ_1 and μ_2.

From ① $\Rightarrow 1 - \mu_1 = 2\mu_2 x$

$$\Rightarrow \boxed{x = \frac{1-\mu_1}{2\mu_2}}$$

From ② $-\mu_1 - 2\mu_2 y = 0$

$$\Rightarrow \mu_1 + 2\mu_2 y = 0 \Rightarrow \boxed{y = \frac{-\mu_1}{2\mu_2}}$$

7

(c) Write z in terms of μ_2 and solve the conditions.

From 4 $-(x+y+z-1)=0$

$$\Rightarrow x+y+z-1=0$$

$$\Rightarrow \frac{1-\mu_1}{2\mu_2}-\frac{\mu_1}{2\mu_2}+z-1=0$$

$$\Rightarrow \frac{1-\mu_1-\mu_1}{2\mu_2}+z-1=0$$

$$z=1-\left(\frac{1-2\mu_1}{2\mu_2}\right)$$

From 3

$$2-\mu_1-\mu_2=0 \Rightarrow \cancel{\mu_2} \; 2-\mu_2=\mu_1$$

$$\Rightarrow \cancel{z=1-2}$$

$$z=1-\left(\frac{1-2(2-\mu_2)}{2\mu_2}\right)$$

$$z=1-\frac{1-4+2\mu_2}{2\mu_2} \Rightarrow z=1-\left(\frac{-3}{2\mu_2}+1\right)$$

$$\cancel{z=1-\left(\frac{-3}{2\mu_2}+1\right) \Rightarrow z=\frac{-3}{2x}}$$

$$\cancel{z=2}$$

$$\Rightarrow z=\cancel{1}+\frac{3}{2\mu_2}-\cancel{1}$$

$$\boxed{z=\frac{3}{2\mu_2}}$$

From 5 $-\left(x^2+y^2+z-\frac{7}{4}\right)=0$

$$\cancel{-\left(\left(\frac{1-\mu_1}{2\mu_2}\right)^2+\left(\frac{-\mu_1}{2\mu_2}\right)^2+\frac{3}{2\mu_2}-\frac{7}{4}\right)=0}$$

8

$$-\left[\left(\frac{1-(2-\mu_2)}{2\mu_2}\right)^2+\left(\frac{-\mu_1}{2\mu_2}\right)^2+\frac{3}{2\mu_2}-\frac{7}{4}\right]=0$$

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Question 4 Let $f(x, y) = xy + x + y$. Consider the problem of maximizing $f(x, y)$ subject to the constraints $x^2 + y^2 \leq 2$, $x + y \leq 1$.

(a) Setup the Lagrangian and check the NDCQ condition.

$g_1(x,y) = x^2+y^2 \leq 2$, $\quad g_2(x,y) = x+y \leq 1$

$\Rightarrow Dg(x,y) = \begin{pmatrix} 2x & 2y \\ 1 & 1 \end{pmatrix} \Rightarrow$ the rank of $Dg(x,y)$ is less than $m=2$ iff $x=y=0 \Rightarrow$ NDCQ satisfied.

3

$L(x,y,\lambda_1,\lambda_2) = xy + x + y - \lambda_1(x^2+y^2-2) - \lambda_2(x+y-1)$

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(b) Write down the first order conditions. Then show that we must have $y = x$.

① $\frac{\partial L}{\partial x} = y+1 - 2\lambda_1 x - \lambda_2 = 0$

② $\frac{\partial L}{\partial y} = x+1 - 2\lambda_1 y - \lambda_2 = 0$ 4

+2

③ $\lambda_1 \geq 0$, $\lambda_2 \geq 0$

④ $\lambda_1[x^2+y^2-2] = 0$, $\lambda_2[x+y-1] = 0$

⑤ $x^2+y^2 \leq 2$, $x+y \leq 1$

$\Rightarrow$ let $\lambda_1 = 0 = \lambda_2 \Rightarrow y+1=0$ and ~~x+1~~ $x+1=0$

$\Rightarrow y=-1=x \Rightarrow y=x$

7+3

(c) Solve the set of first order conditions for $(x, y, \lambda_1, \lambda_2)$. Hint: consider all possible cases about the multipliers.

① $\lambda_1 = 0,\ \lambda_2 = 0 \Rightarrow x = y = -1$

$\Rightarrow (-1, -1, 0, 0)$ ---- (*)

② $\lambda_1 \neq 0,\ \lambda_2 = 0$

$y + 1 - 2\lambda_1 x = 0 \Rightarrow y - 2\lambda x = -1$

$x + 1 - 2\lambda_1 y = 0 \qquad x - 2\lambda_1 y = -1$

$\Rightarrow y - 2\lambda_1 x = x - 2\lambda_1 y$

$\Rightarrow \cancel{y - x - 2\lambda_1 x + 2\lambda_1 y = 0}$

$y + 2\lambda_1 y = x + 2\lambda_1 x$

$y(1 + 2\lambda_1) = x(1 + 2\lambda_1)$ ←

$y = x$

$\Rightarrow \lambda_1 (x^2 + x^2 - 2) = 0 \Rightarrow x^2 + x^2 - 2 = 0$

$2x^2 = 2 \Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

$\Rightarrow y = x = \pm 1 \Rightarrow 1 + 1 - 2\lambda_1 = 0$

$2 = 2\lambda_1 \Rightarrow \lambda_1 = 1$

$\Rightarrow (1, 1, 1, 0)$ and $(-1, -1, 0, 0)$ --- (*)

2

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Question 3 Prove the following:

5 1. Let α and β be positive constants and f, g are convex functions on a convex set $U \subset \mathbb{R}^n$. Show that $\alpha f + \beta g$ is a convex functions on U.

Let $h(\vec{x}) = \alpha f(\vec{x}) + \beta g(\vec{x})$

$h(t\vec{x}+(1-t)\vec{y}) = \alpha f(t\vec{x}+(1-t)\vec{y}) + \beta g(t\vec{x}+(1-t)\vec{y})$

$\leqslant \alpha\left[t f(\vec{x}) + (1-t) f(\vec{y})\right] + \beta\left[t g(\vec{x}) + (1-t) g(\vec{y})\right]$

$= t\left[\alpha f(\vec{x}) + \beta g(\vec{x})\right] + (1-t)\left[\alpha f(\vec{y}) + \beta g(\vec{y})\right]$

$= t\, h(\vec{x}) + (1-t)\, h(\vec{y})$

$\Rightarrow$ h is convex.

4 2. Let f be a quasiconcave, positive function defined over the convex subset S of $\mathbb{R}^n$. Show that $g(\mathrm{x}) = 1/f(\mathrm{x})$ is a quasiconvex function over S.

since $f(\vec{x})$ is quasiconcave then the set

$C^{+}_{x_0} = \{\vec{x} \mid f(\vec{x}) \geq f(\vec{x}_0)\}$ is a convex set

$= \{\vec{x} \mid \frac{1}{f(\vec{x})} \leqslant \frac{1}{f(\vec{x}_0)}\}$ is convex set

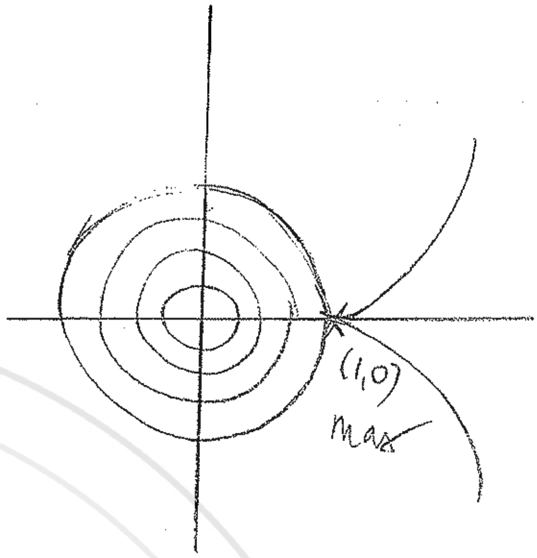
$= \{\vec{x} \mid g(\vec{x}) \leq g(\vec{x}_0)\}$ is convex set.

$\therefore$ g is quasi convex.

Question 4 Consider the problem of maximizing $f(x, y) = -(x^2 + y^2)$ subject to the constraint $y^2 - (x-1)^3 = 0$.

(a) Solve the above problem graphically.

$y = \pm (x-1)^{3/2}$



maximum at $(1,0)$

$f(1,0) = -1$

(b) Solve the problem by including a multiplier μ_0 for the objective function.

$L(x, y, \mu_0, \mu_1) = -\mu_0 (x^2 + y^2) - \mu_1 (y^2 - (x-1)^3)$

(1) $-2\mu_0 x + 3\mu_1 (x-1)^2 = 0$

(2) $-2\mu_0 y - 2\mu_1 y = 0$

(3) $y^2 - (x-1)^3 = 0$

(4) $\mu_0 = 0$ or 1.

(5) $(\mu_0, \mu_1) \neq (0,0)$

If $\mu_0 = 0 \overset{(1)}{\Longrightarrow} 3\mu_1 (x-1)^2 = 0$, $\mu_1 \neq 0 \Rightarrow x = 1$

$(3) \Rightarrow y^2 = 0 \Rightarrow y = 0$, μ_1 any number $\neq 0$.

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Question 5 Determine whether the following functions are quasiconvex, quasiconcave, or neither

10

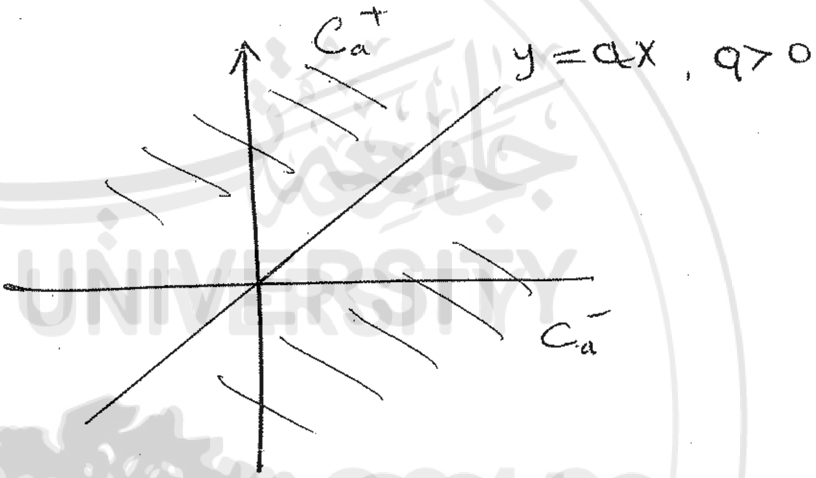
3 (a) $f(x, y, z) = \ln(x^2 + y^2 + z^2)$.

$x^2 + y^2 + z^2$ is convex function.

$\ln z$ is a M.T.

$\Rightarrow f$ is q-convex (M.T. of convex).

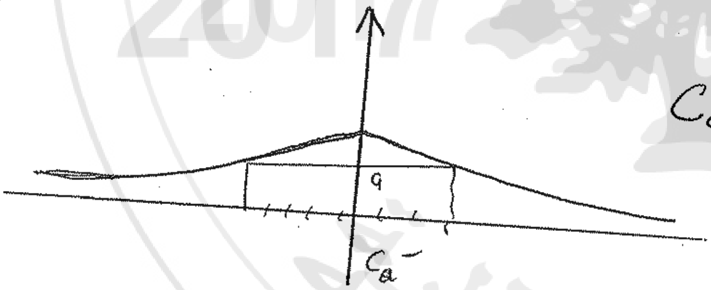
3 (b) $f(x, y) = \frac{y}{x}, x > 0$.



$\frac{y}{x} = a \Rightarrow y = aX$

Since both C_a^+, C_a^- are convex sets $\Rightarrow f$ is both.

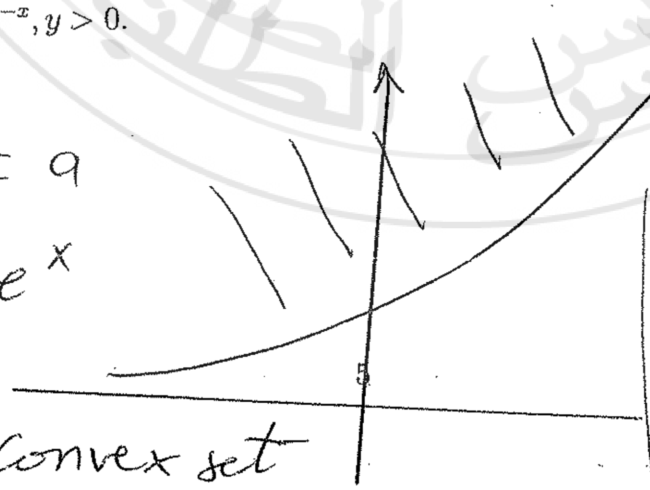
2 (c) $f(x) = e^{-|x|}$.



C_a^+ is a convex set

$\Rightarrow$ q-concave.

2 (d) $f(x, y) = ye^{-x}, y > 0$.



$y = ae^x$

$ye^{-x} = a$

$y = ae^x$

C_a^+ is convex set

$\Rightarrow$ q-Concave.

$$\begin{vmatrix} 0 & -ye^{-x} & e^{-x} \\ -ye^{-x} & ye^{-x} & -e^{-x} \\ e^{-x} & -e^{-x} & 0 \end{vmatrix}$$

$= ye^{-x}(e^{-2x}) - e^{-x}(ye^{-2x} - ye^{-2x})$

$= ye^{-3x} > 0 \Rightarrow$ q concave

10 **Question 6** Answer the following

3 (a) Is having decreasing marginal utility, $\frac{\partial^2 U}{\partial x_i^2} < 0$, for all i, an ordinal property? Why?

let $f(\vec{x}) = g(U(\vec{x}))$, g is a M.T.

$$\frac{\partial f}{\partial x_i} = g'(U(\vec{x}))\,\frac{\partial U}{\partial x_i}$$

$$\frac{\partial^2 f}{\partial x_i^2} = \underset{+}{g'(U(\vec{x}))}\,\underset{-}{\frac{\partial^2 U}{\partial x_i^2}} + \underset{??}{g''(U(\vec{x}))}\,\underset{+}{\left(\frac{\partial U}{\partial x_i}\right)^2}$$

$\therefore$ This is not an ordinal property.

(b) Show that if f is concave on an open convex set $U \subset \mathbb{R}^n$ and $Df(x^*) = 0$ then $f(x^*)$ is a global maximum.

3

f is concave iff $f(\vec{y}) - f(\vec{x}) \leq Df(\vec{x})(\vec{y}-\vec{x})$

$\forall$ $Df(\vec{x}^*) = \vec{0} \Rightarrow f(\vec{y}) - f(\vec{x}^*) \leq \vec{0}$

$\Rightarrow f(\vec{y}) \leq f(\vec{x}^*)$ $\forall \vec{y} \in U \Rightarrow f(\vec{x}^*)$ is global max

4 (c) Find the value(s) of c such that the function $f(x, y) = x^2 + 2cxy + y^2$ is convex.

$$D^2 f(x,y) = \begin{pmatrix} 2 & 2c \\ 2c & 2 \end{pmatrix}$$

?? $D^2 f(x,y)$ is +ve semidef.

1^{st} order PMs : $2 > 0$, $2 > 0$

$$|D^2 f| = 4 - 4c^2 \geq 0 \iff c^2 \leq 1 \iff -1 \leq c \leq 1.$$

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