Question 1:

Second Divided Differences:

$$f[x_0, x_1, x_2] = \frac{\left(\frac{y_2 - y_1}{x_2 - x_1}\right) - \left(\frac{y_1 - y_0}{x_1 - x_0}\right)}{(x_2 - x_0)}$$

$$f[1, -2, 0] = \frac{\left(\frac{4 - 3}{0 - -2}\right) - \left(\frac{3 - 2}{-2 - 1}\right)}{(0 - 1)}$$

$$f[1, -2, 0] = \frac{\left(0.5 - -\left(\frac{1}{3}\right)\right)}{-1} = -0.833$$

Question 2:

$x = g_1(x, y, z) = \cos(xy)$

$y = g_2(x, y, z) = e^{y+x+z}$

$z = g_3(x, y, z) = x^2 - yz$

Initial guess: $(x_0, y_0, z_0) = (0, 1, -1)$

n	p_n	q_n	r_n
1	$p_1 = g_1(0,1,-1) = \cos(0) = 1$	$q_1 = g_2(0,1,-1) = e^0 = 1$	$r_1 = g_3(0,1,-1) = 0 + 1 = 1$
2	$p_2 = g_2(1,1,1) = \cos(1) = 0.540$	$q_2 = g_2(0.54,1,-1) = e^{0.54} = 1.72$	$r_2 = g_3(0.54,1.72,-1) = 2.01$

The answer $= (x_2, y_2, z_2) = (0.54, 1.72, 2.01)$

Question 3:

$f(x, y) = 4x^2 + 3y^2 - 16$

$g(x, y) = xy - 2$

Newton's iteration: $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} - J(x_{n-1}, y_{n-1}) \begin{pmatrix} f(x_{n-1}, y_{n-1}) \\ g(x_{n-1}, y_{n-1}) \end{pmatrix}$

To find Jacobian:

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

$$J(x,y) = \begin{pmatrix} 8x & 6y \\ y & x \end{pmatrix}$$

$$\text{First Iteration} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J(x_0, y_0) \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix}$$

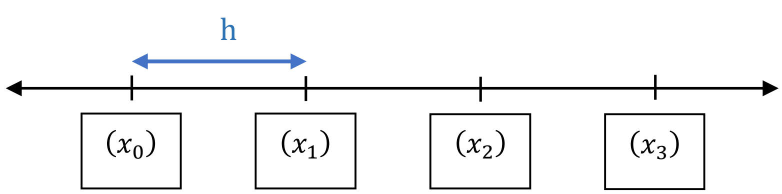
$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - J(0,1) \begin{pmatrix} f(0,1) \\ g(0,1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -13 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -12 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -12 \end{pmatrix} - J(-2,-12) \begin{pmatrix} f(-2,-12) \\ g(-2,-12) \end{pmatrix} = \begin{pmatrix} -2 \\ -12 \end{pmatrix} - \begin{pmatrix} -16 & -72 \\ -12 & -2 \end{pmatrix} \begin{pmatrix} 432 \\ 22 \end{pmatrix} = \begin{pmatrix} 8500 \\ 5220 \end{pmatrix}$$

Question 4:

$$p_3(x) = \sum_{k=0}^{3} \left(y_k * \mathrm{L}_{n,k}(x) \right)$$

➔ We are using uniform partition so we need to find h (we have 4 points)



$$h = \frac{(x_3 - x_0)}{3} = \frac{4-1}{3} = 1$$

a) Lagrange polynomial:

$$p_3(2.5) = \sum_{k=0}^{3} \left(y_k * \mathrm{L}_{n,k}(x) \right)$$

$$= \left(y_0 * \mathrm{L}_{3,0}(2.5) \right) + \left(y_1 * \mathrm{L}_{3,1}(2.5) \right) + \left(y_2 * \mathrm{L}_{3,2}(2.5) \right) + \left(y_3 * \mathrm{L}_{3,3}(2.5) \right)$$

Points	y	$L_{n,k}(x)$	$y_k * L_{n,k}(x)$
x_0	$f(x) = ln(1+2) = ln3 = 1.10$	$L_{3,0}(2.5) = \frac{(2.5-2)(2.5-3)(2.5-4)}{(1-2)(1-3)(1-4)} = -0.0625$	-0.0688
x_1	$f(x) = ln(2+2) = ln4 = 1.39$	$L_{3,1}(2.5) = \frac{(2.5-1)(2.5-3)(2.5-4)}{(2-1)(2-3)(2-4)} = 0.563$	0.783
x_2	$f(x) = ln(3+2) = ln5 = 1.61$	$L_{3,2}(2.5) = \frac{(2.5-1)(2.5-2)(2.5-4)}{(3-1)(3-2)(3-4)} = 0.563$	0.906
x_3	$f(x) = ln(4+2) = ln6 = 1.79$	$L_{3,3}(2.5) = \frac{(2.5-1)(2.5-2)(2.5-3)}{(4-1)(4-2)(4-3)} = -0.0625$	-0.112

→Taking Values from the table:

$p_3(2.5) = -0.0688 + 0.783 + 0.906 + -0.112 = 1.508$

b) Newton Polynomial:

$p_3(x) = a_0 + a_1(x-x_0) + a_1(x-x_0)(x-x_1) + a_2(x-x_0)(x-x_1)(x-x_2)$

$+a_3(x-x_0)(x-x_1)(x-x_2)(x-x_3)$ =1.57

a_n		$a_n = (x - x_n)$
a_0	$a_0 = y_0 = 1.10$	1.10
a_1	$a_1 = \frac{(y_1-y_0)}{(x_1-x_0)} = \frac{(1.39-1.10)}{(2-1)} = 0.29$	0.435
a_2	$a_2 = \frac{\frac{(y_2-y_1)}{(x_2-x_1)} - \frac{(y_1-y_0)}{(x_1-x_0)}}{(x_2-x_0)} = 0.11$	0.0825
a_3	$a_3 = \frac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3-x_0}$ $= \frac{\frac{(y_3-y_2)}{(x_3-x_2)} - \frac{(y_2-y_1)}{(x_2-x_1)}}{(x_3-x_1)} - \frac{\frac{(y_2-y_1)}{(x_2-x_1)} - \frac{(y_1-y_0)}{(x_1-x_0)}}{(x_2-x_0)} = -0.13$	-0.0488

c) Upper bound for the error

$$|E_3| \leq \frac{h^4 * M_4}{24}$$

$h^4 = 1$

➔ We need to find $f(x)^4$

$f(x)^1 = \frac{1}{x+2}$

$f(x)^2 = -\frac{1}{(x+2)^2}$

$f(x)^3 = \frac{2}{(x+2)^3}$

$f(x)^4 = -\frac{6}{(x+2)^4}$

$f(x)^4$ *is decreasing so the maximum value is* $f(4) = 4.63 * 10^{-3}$

$$|E_3| \leq \frac{4.63 * 10^{-3}}{24} \leq 1.93 * 10^{-4}$$

Question 5

a) We need to find the third derivative of the function and show it doesn't equal zero at x=1 (root p=1)

$$f'(x) = (x-1)^2 * \frac{1}{x} + \ln(x) * 2 * (x-1)$$

$$f'(1) = (1-1)^2 * \frac{1}{1} + \ln(1) * 2 * (1-1) = 0$$

$$f''(x) = (x-1)^2 * \frac{-1}{x^2} + \frac{1}{x} * 2 * (x-1) + \ln(x) * 2 + 2 * (x-1) * \frac{1}{x}$$

$$f''(1) = (1-1)^2 * \frac{-1}{1^2} + \frac{1}{1} * 2 * (1-1) + \ln(1) * 2 + 2 * (1-1) * \frac{1}{1} = 0$$

$$f'''(x) = (x-1)^2 * \frac{2}{x^3} + \frac{-1}{x^2} * 2 * (x-1) + \frac{1}{x} * 2 + 2 * (x-1) * \frac{-1}{x^2} + \frac{2}{x} + 2 * (x-1) * \frac{-1}{x^2} + \frac{2}{x}$$

$$f'''(1) = (1-1)^2 * \frac{2}{1^3} + \frac{-1}{1^2} * 2 * (1-1) + \frac{1}{1} * 2 + 2 * (1-1) * \frac{-1}{1^2} + \frac{2}{1} + 2 * (1-1) * \frac{-1}{1^2} + \frac{2}{1} = 2 + 2 + 2 = 6$$

Since $f'''(x) \neq 0\ so\ M = 3$ (cubic root)

b) Order of convergence (R)= 1 (since the root is a multiple root)

c) Asymptotic error constant (A)=$\frac{M-1}{M} = \frac{2}{3}$

Question 6:

$f(x) = x^2 - 1$

a) Secant method of approximation:

$$P_{n+2} = P_{n+1} - \left(\frac{P_{n+1} - P_n}{f(P_{n+1}) - f(P_n)}\right) * f(P_{n+1})$$

$$P_2 = P_1 - \left(\frac{P_1 - P_0}{f(P_1) - f(P_0)}\right) * f(P_1) = 2 - \left(\frac{2-3}{f(2) - f(3)}\right) * f(2)$$
$$= 2 - \left(\frac{-1}{(4-1) - (9-1)}\right) * (4-1) = 2 - \left(\frac{-3}{-5}\right) = 2 - 0.6 = 1.4$$

$$P_3 = P_2 - \left(\frac{P_2 - P_1}{f(P_2) - f(P_1)}\right) * f(P_2) = 1.4 - \left(\frac{1.4 - 2}{f(1.4) - f(2)}\right) * f(1.4)$$
$$= 1.4 - \left(\frac{-0.6}{(1.96 - 1) - (4-1)}\right) * 0.96 = 1.4 - \left(\frac{-0.6}{-2.04}\right) * 0.96$$
$$= 1.4 - (0.294 * 0.96) = 1.4 - 0.282 = 1.12$$

b) Order of Convergence (R)=

→We need to find the derivative at which $f^M(P) \neq 0$

$f'(x) = 2 * x$

$f'(1) = 2 * 1 = 2$

so $M = 1$ and so $R = 1.618$

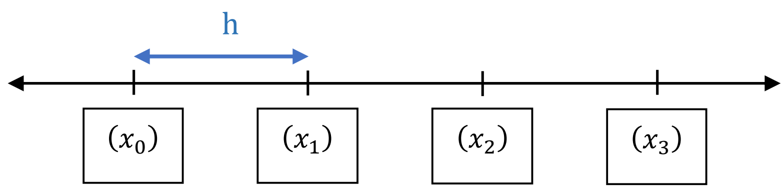
c) Asymptotic error constant:

$$A = \left|\frac{f''(p)}{2 * f'(p)}\right|^{0.618} = \left|\frac{2}{2 * 2}\right|^{0.618} = 0.5^{0.618} = 0.652$$

Question 7:

$f(x) = e^x$

Uniform partition:



$$h = \frac{4 - 1}{3} = 1$$

Points	y
x_0	$y_0 = f(x) = e^1 = 2.72$
x_1	$y_1 = f(x) = e^2 = 7.39$
x_2	$y_2 = f(x) = e^3 = 20.1$
x_3	$y_3 = f(x) = e^4 = 54.6$

Lagrange polynomial of order 1:

$$p_1(x) = \sum_{k=0}^{1}\left(y_k * \mathrm{L}_{n,k}(x)\right)$$

$$= \left(y_0 * \mathrm{L}_{1,0}(x)\right) + \left(y_1 * \mathrm{L}_{1,1}(x)\right) = \left(2.72 * \frac{(x-2)}{(1-2)}\right) + \left(7.39 * \frac{(x-1)}{(2-1)}\right)$$

$$= -2.72x + 5.44 + 7.39x - 7.39 = 4.67x - 1.95$$

$$p_2(x) = \sum_{k=0}^{2}\left(y_k * \mathrm{L}_{n,k}(x)\right)$$

$$= \left(y_0 * \mathrm{L}_{2,0}(x)\right) + \left(y_1 * \mathrm{L}_{2,1}(x)\right) + \left(y_0 * \mathrm{L}_{2,2}(x)\right)$$

$$= y_0 * \left(\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}\right) + y_1 * \left(\frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}\right) + y_2 * \left(\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\right)$$

$$= 2.72 * \frac{(x - 2)(x - 3)}{(1 - 2)(1 - 3)} + 7.39 * \left(\frac{(x - 1)(x - 3)}{(2 - 1)(1 - 2)}\right) + 20.1 * \left(\frac{(x - 1)(x - 2)}{(3 - 1)(3 - 2)}\right)$$

$$= 1.36 * (x^2 - 5x + 6) - 7.39 * (x^2 - 4x + 3) + 10.1 * (x^2 - 3x + 2)$$

$$= 4.07x^2 - 7.54x + 6.19$$

$$p_3(x) = \sum_{k=0}^{3} \left(y_k * \mathrm{L}_{n,k}(x)\right)$$

$$= \left(y_0 * \mathrm{L}_{3,0}(x)\right) + \left(y_1 * \mathrm{L}_{3,1}(x)\right) + \left(y_2 * \mathrm{L}_{3,2}(x)\right) + \left(y_3 * \mathrm{L}_{3,3}(x)\right)$$

$$= y_0 * \left(\frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}\right) + y_1 * \left(\frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}\right) + y_2 * \left(\frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}\right) + y_3 * \left(\frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}\right)$$

$$= 0.453 * \left((x - 2)(x - 3)(x - 4)\right) + 3.7 * \left((x - 1)(x - 3)(x - 4)\right) - 10.1 * \left((x - 1)(x - 2)(x - 4)\right) + 9.1\left((x - 1)(x - 2)(x - 3)\right)$$

$$= 0.453 * (x^3 - 9x^2 + 26x - 24) + 3.7 * (x^3 - 8x^2 + 19x - 12) - 1.01 * (x^3 - 7x^2 + 14x - 8) + 9.1 * (x^3 - 6x^2 + 11x - 6)$$

$$=12.2x^3 - 81.2x^2 + 168x - 12.9$$